Ornstein-Uhlenbeck Processes Time Changed with Additive Subordinators and Their Applications in Commodity Derivative Models

Lingfei Li[*]   Rafael Mendoza-Arriaga[†]

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Abstract

We characterize Ornstein-Uhlenbeck (OU) processes time-changed with additive subordinators as time-inhomogeneous Markov semimartingales, based on which a new class of commodity derivative models is developed. Analytical solutions are obtained for European and Bermudan futures options via eigenfunction expansions, with American option prices computed efficiently via extrapolation. Calibration examples show that these new models can be better alternatives than those developed in Li and Linetsky (2012) by being much more parsimonious and faster for option pricing while calibrating very well to implied volatilities across maturities. Our method can be easily applied to many other processes popular in a variety of areas besides finance to develop time-inhomogeneous Markov processes with desirable features and tractability.

Key Words: commodity derivatives, additive processes, time change, Ornstein-Uhlenbeck, time-dependent and mean-reverting jumps, eigenfunction expansions.

1   Introduction

Li and Linetsky (2012) developed a class of commodity derivative models where under the risk-neutral measure chosen by the market, the spot price S_t is assumed to follow

$$S_t = F(0,t)e^{X_t^\phi - G(t)}, \ X_0^\phi = x_0, \tag{1.1}$$

where $\{F(0,t), t \geq 0\}$ is the futures curve observed from the market at time 0, and $G(t)$ is a function of time such that $\mathbb{E}[S_t] = F(0,t)$ under the risk-neutral measure. X^ϕ is a *subordinate OU (SubOU) process* defined by $X_t^\phi := X_{\mathcal{T}_t}$, where X is an OU diffusion, i.e., $dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$, $X_0 = x_0$ ($\kappa, \sigma > 0$, B: a standard Brownian motion), and $\mathcal{T}$ is a Lévy subordinator (nonnegative Lévy process with $\mathcal{T}_0 = 0$), independent of X. The Laplace transform of $\mathcal{T}$ is well known and given by (e.g., Sato (1999))

$$E[e^{-\lambda \mathcal{T}_t}] = e^{-\phi(\lambda)t}, \ \phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda\tau}\right)\nu(d\tau), \ \gamma \geq 0, \ \int_{(0,\infty)} (\tau \wedge 1)\nu(d\tau) < \infty.$$

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[*]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong. Email: lfli@se.cuhk.edu.hk.

[†]Department of Information, Risk and Operations Management, McCombs School of Business, The University of Texas at Austin. Email: rafael.mendoza-arriaga@mccombs.utexas.edu.

Li and Linetsky (2012) characterized X^ϕ as a time-homogeneous Markov semimartingale and explicitly found its characteristics and sample path decomposition. In general, X^ϕ is a jump-diffusion (if $\gamma > 0$) or pure jump (if $\gamma = 0$). Furthermore, its jump measure is state-dependent and mean-reverting, making it ideal for modeling commodities. Using the eigenfunction expansion of the transition semigroup of X^ϕ, Li and Linetsky (2012) obtained analytical formulas for European futures options. Li and Linetsky (2013) extended this approach and obtained analytical formulas for Bermudan options with American option prices computed efficiently via Richardson extrapolation.

SubOU-based models are mean-reverting commodity market counterparts of exponential Lévy models for equities. Just as the exponential Lévy models, they can calibrate to a single maturity implied volatility smile very well. However, these models have difficulties calibrating across maturities (see Li and Linetsky (2012)). This is due to the time-dependent behavior of commodity prices and the slower decay rate of the smile effect than that implied by SubOU processes. This difficulty is attenuated in Li and Linetsky (2012) by further time-changing X^ϕ with an independent absolutely continuous time change $A_t := \int_0^t (a(u) + Z_u) du$, and replacing X^ϕ by $Y := X_{A_t}^\phi$ in (1.1) (a similar approach is used by Carr et al. (2003) to improve the calibration performance of Lévy-based equity models). We refer to the process Y as the *time-dependent stochastic volatility SubOU (TDSVSubOU) process*. Here $a(\cdot)$ is a deterministic function of time to model time dependency, and Z is a CIR process to model stochastic volatility. Correspondingly, the TDSVSubOU-based models are the counterpart of Lévy models with stochastic volatility (e.g., Carr et al. (2003)). These models are able to calibrate to multiple maturities simultaneously. However, they are heavily parameterized and computationally challenging. For instance, compared to SubOU models, TDSVSubOU models require at least four extra parameters for the CIR process, in addition to the parameters required for the function $a(\cdot)$. Also, since the process Y is not Markov, but the bivariate process (Y, Z) is, one needs to deal with an additional dimension, which makes it very difficult to compute Bermudan and American option prices. Li and Linetsky (2012) managed to get an analytical formula for European options. However, computing it is quite time consuming.

Therefore, we are led to the following question: can we find a more parsimonious model which is tractable and fast for option pricing and at the same time achieves good calibration results to multiple maturities? Our solution is to time change the OU diffusion by an additive subordinator, which is a nonnegative additive process (additive processes are Lévy process without the stationary increment requirement; see Sato (1999)). The resulting process is named as *additive subordinate OU (ASubOU) process*. We characterize it as a Markov semimartingale, and explicitly give its local characteristics and sample path decomposition. Compared to SubOU processes, ASubOU processes in general are time-inhomogeneous. In particular, the jump measure is time-dependent in addition to being state-dependent and mean-reverting. This provides more flexibility for calibration. Under ASubOU processes, pricing European, Bermudan and American options is just as easy as the SubOU case using eigenfunction expansions. The only change is to replace the Laplace transform of a Lévy subordinator by that of an additive subordinator.

In practice, our modeling framework is very flexible as there is a great variety of additive subordinators to choose from. We present some examples in section 4. In our numerical experiments we revisit two calibration cases considered in Li and Linetsky (2012) and use the Inverse Gaussian-Sato subordinator. This specification is much more parsimonious than the TDSVSubOU model as it only requires one more parameter compared to the SubOU model. Our results show that under this simple specification, the ASubOU model already calibrates almost as well to multiple maturities as the TDSVSubOU model. Hence, from the perspective of parsimony and tractability, the ASubOU model can be better alternatives than the TDSVSubOU model.

While Lévy subordination (i.e., time changing by Lévy subordinators) is already a standard

technique in probability for building new processes, additive subordination is new in the litera-ture. To the best of our knowledge, Mijatovic and Pistorius (2010) is the only paper so far on additive subordination of Markov processes, which extended Phillips' theorem Phillips (1952) on Lévy subordination to additive subordination, while our paper is the first one which develops new financial models with desirable features by additive subordination of a non-Lévy process and also obtains tractability for pricing European, Bermudan and American options. We give Markov char-acterization of ASubOU processes based on Mijatovic and Pistorius (2010), and further derive semimartingale characterization and eigenfunction expansions of the transition operator. Based on these results we are able to understand the sample paths of ASubOU processes and price options.

The eigenfunction expansion approach is developed by Davydov and Linetsky (2003) to price European options for scalar diffusions and extended by Li and Linetsky (2013) to options with early exercise for general subordinate diffusions, all of which are time-homogeneous Markov processes. This paper provides an example of how to extend this approach to time-inhomogeneous ones. In general starting with a time-homogeneous Markov process with known eigenfunction expansions of the transition operator, one can easily incorporate time dependency by additive subordination while *retaining tractability* by eigenfunction expansions. Our results shed light on the usefulness of addi-tive subordination as a tool for transforming time-homogenous processes into time-inhomogeneous ones. We anticipate applications to be found in other areas besides finance. For example, the transition operator of many diffusions and birth-and-death processes popular in a variety of areas such as ecology, genetics, queueing, etc., can be expressed in terms of eigenfunction expansions (c.f.Karlin and McGregor (1957); Karlin and Taylor (1981); Linetsky (2008); Schoutens (2000)). Time changing these processes by additive subordinators generate time-inhomogeneous versions with tractability. To study their sample paths, our method for ASubOU processes can be readily generalized and applied.

2 ASubOU Processes as Time-inhomogeneous Markov Semimartin-gales

We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting an OU diffusion X as defined in section 1, and an additive subordinator T, independent of X. The Laplace transform of T is given by (see e.g.Jacod and Shiryaev (2003))

$$\mathbb{E}[e^{-\lambda(T_t - T_s)}] = e^{-\int_s^t \psi(\lambda, u) du}, \quad \text{with} \quad \psi(\lambda, u) = \lambda \gamma(u) + \int_{(0,\infty)} \left(1 - e^{-\lambda\tau}\right) \nu(u, d\tau).$$

where for all $u \geq 0$, $\gamma(u) \geq 0$, $\int_{(0,\infty)} (\tau \wedge 1)\nu(u, d\tau) < \infty$, and $\int_0^t (\gamma(u) + \int_{(0,\infty)} (\tau \wedge 1)\nu(u, d\tau)) du < \infty$ for all $t > 0$. Let $(q_{s,t})_{0 \leq s \leq t}$ be the family of transition probability measures of T.

The *ASubOU process* $(X_t^\psi)_{t \geq 0}$ is constructed as $X_t^\psi := X_{T_t}$ with $X_0^\psi = x_0$ (we write ψ in the superscript to signify the Laplace exponent of T is ψ). Let $\mathcal{F}_t^0 := \sigma(X_u^\psi : 0 \leq u \leq t)$ be the natural filtration generated by X^ψ, and $\mathcal{F}_t := \bigcap_{u \geq t} \mathcal{F}_u^0$ be the right-continuous version of $\mathcal{F}_t^0$. It is easy to see that the sample paths of X^ψ are càdlàg, and that, in general, if T is an additive (resp. a Lévy) subordinator then X^ψ is a time-inhomogeneous (resp. time-homogeneous) process. In order to facilitate our analysis of the ASubOU process X^ψ, in what follows, we use the associated time-space process $(s + t, X_{s+t}^\psi)_{t \geq 0}$ and define $E := \mathbb{R}_+ \times \mathbb{R}$. First we have the following Markov characterization.

Theorem 1. *For any $s \geq 0$, $(s + t, X_{s+t}^\psi)_{t \geq 0}$ is a time-homogenous Markov (in fact Feller) process*

w.r.t. $(\mathcal{F}_{s+t})_{t\geq 0}$. *Denote by* $\mathcal{G}^{\psi}$ *its infinitesimal generator. Then for* $f \in C_c^{1,2}(E) \subseteq \mathcal{D}(\mathcal{G}^{\psi})$

$$\mathcal{G}^{\psi}f(s,x) = \frac{\partial f}{\partial s}(s,x) + \frac{1}{2}\gamma(s)\sigma^2\frac{\partial^2 f}{\partial x^2}f(s,x) + b(s,x)\frac{\partial f}{\partial x}(s,x)$$
$$+ \int_{y\neq 0}\left(f(s,x+y) - f(s,x) - y\mathbf{1}_{\{|y|\leq 1\}}\frac{\partial f}{\partial x}(s,x)\right)\Pi(s,x,dy), \qquad (2.2)$$

where $b(s,x) = \gamma(s)\kappa(\theta - x) + \int_{(0,\infty)}\left(\int_{\{|y|\leq 1\}} yp(\tau,x,x+y)dy\right)\nu(s,d\tau)$, *and the time- and state-dependent jump measure* $\Pi(s,x,dy) = \pi(s,x,y)dy$ *having density defined for all* $y \neq 0$, *and*

$$\pi(s,x,y) = \int_{(0,\infty)} p(\tau,x,x+y)\nu(s,d\tau),$$

where $p(\tau,x,x+y)$ *is the OU diffusion transition density from* x *to* $x+y$ *after* τ *units of time. Furthermore,* $\Pi(s,x,dy)$ *is a Lévy-type measure, i.e.* $\int_{|y|\neq 0}(y^2 \wedge 1)\Pi(s,x,dy) < \infty$.

Proof. Since the OU diffusion X is Feller, Thm.1 in Mijatovic and Pistorius (2010) shows $(s + t, X_{s+t}^{\psi})$ is Markov and Feller w.r.t. $(\mathcal{F}_t^0)_{t\geq 0}$. The Feller property allows us to further deduce that it must also be Markov w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ ((Revuz and Yor, 1999, Prop.2.14)). (Mijatovic and Pistorius, 2010, Thm.1) also shows for $f \in C_c^{1,2}(E)$, $\mathcal{G}^{\psi}$ is given by $(f_s(x) := f(s,x))$

$$\mathcal{G}^{\psi}f(s,x) = \frac{\partial f}{\partial s}(s,x) + \gamma(s)\mathcal{G}f_s(x) + \int_{(0,\infty)}(\mathcal{P}_{\tau}f_s(x) - f_s(x))\nu(s,d\tau). \qquad (2.3)$$

where $\mathcal{G}$ *and* $(\mathcal{P}_t)_{t\geq 0}$ *are the infinitesimal generator and transition semigroup of the OU diffusion* X, *and* $\mathcal{G}f(x) = \frac{1}{2}\sigma^2\frac{d^2 f}{dx^2}(x) + \kappa(\theta - x)\frac{df}{dx}(x)$ *for* $f \in C_c^2(\mathbb{R})$. *For each* $(s,x) \in E$ *we can write*

$$(\mathcal{P}_{\tau}f_s - f_s)(x) = \int_{\mathbb{R}}\left(f(s,x+y) - f(s,x) - \mathbf{1}_{\{|y|\leq 1\}}y\frac{\partial f}{\partial x}(s,x)\right)p(\tau,x,x+y)dy$$
$$+ \left(\int_{\{|y|\leq 1\}}yp(\tau,x,x+y)dy\right)\frac{\partial f}{\partial x}(s,x). \qquad (2.4)$$

To prove (2.2), we point out several useful facts. From (McKean, 1956, Thm.4.5), $p(\tau,x,x+y)$ satisfies the following as $\tau \to 0$: (a) $\int_{|y|\geq 1}p(\tau,x,x+y)dy \leq C_1\tau$, (b) $\int_{|y|\leq 1}y^2p(\tau,x,x+y)dy \leq C_2\tau$, (c) $\left|\int_{|y|\leq 1}yp(\tau,x,x+y)dy\right| \leq C_3\tau$. $\nu(s,d\tau)$ satisfies (d) $\int_{(0,\infty)}(\tau\wedge 1)\nu(u,d\tau) < \infty$. Now substituting (2.4) into (2.3), and interchanging the integration in y and in τ for the first integral rsults in (2.2). This interchange is justified by Fubini's theorem, by noticing that $|f(s,x+y) - f(s,x) - \mathbf{1}_{\{|y|\leq 1\}}y\frac{\partial f}{\partial x}(s,x)| \leq C(y^2 \wedge 1)$ for some constant $C > 0$, (a), (b) and (d). (c) and (d) ensures the integral in $b(s,x)$ is well defined, and (a) (b) and (d) ensures $\Pi(s,x,dy)$ is a Lévy-type measure. $\square$

We are also able to characterize X^{ψ} as a special semimartingale.

Theorem 2. *(i)* X^{ψ} *is a special semimartingale with the following characteristics* (B,C,ν^{ψ}) *w.r.t to the truncation function* $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$ *(*$b(s,x)$ *and* $\pi(s,x,y)$ *are defined in Theorem 1)*

$$B_t(\omega) = \int_0^t b(s, X_{u-}(\omega))du, \quad C_t(\omega) = \int_0^t \gamma(s)ds \cdot \sigma^2 t, \quad \nu^{\psi}(\omega,ds,dy) = \pi(s, X_{s-}(\omega),y)dsdy.$$

(ii) Denote by μ^{ψ} *the integer-valued random measure associated with the jumps of* X^{ψ} *((Jacod and Shiryaev, 2003, p.69)) and* $X^{\psi,c}$ *the continuous local martingale part of* X^{ψ}. *Then* X^{ψ} *has the*

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following sample path decomposition ($*$ denotes integration w.r.t. a random measure; see (Jacod and Shiryaev, 2003, Chap.II))

$$X_t^\psi(\omega) = x_0 + B_t(\omega) + (x - h(x)) * \nu_t^\psi(\omega) + X_t^{\psi,c}(\omega) + x * (\mu^\psi - \nu^\psi)_t(\omega),$$

with the quadratic variation $[X^{\psi,c}, X^{\psi,c}]_t(\omega) = C_t(\omega)$.

Proof. Step 1. To show X^ψ is a semimartingale with (B, C, ν^ψ) as its characteristics, it suffices to show for every $f(s, x) \in C_b^2(E)$ ((Jacod and Shiryaev, 2003, Thm.II.2.42) and (Schnurr, 2009, Thm.2.26)),

$$
\begin{aligned}
M_t^f := {} & f(t, X_t^\psi) - f(0, X_0^\psi) - \int_0^t \frac{\partial f}{\partial s}(s, X_{s-}^\psi)ds - \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}^\psi)b(s, X_{s-}^\psi)ds \\
& - \frac{1}{2}\sigma^2 t \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-}^\psi)\gamma(s)ds \\
& + \int_{[0,t]\times\{y\neq 0\}} \left[f(s, X_{s-}^\psi + y) - f(X_{s-}^\psi) - \frac{\partial f}{\partial s}(s, X_{s-}^\psi) \right] \pi(s, X_{s-}^\psi, y)dsdy \qquad (2.5)
\end{aligned}
$$

is a local martingale. Choose $\chi(s, x) \in C_c^\infty(E)$ such that $\chi(s, x) = 1$ for $\|(s, x)\| \leq 1$, $\chi(s, x) \leq 1$ for $1 < \|(s, x)\| \leq 2$ and $\chi(s, x) = 0$ for $\|(s, x)\| > 2$. Define $\chi_n(s, x) := \chi(s/n, x/n)$ and $f_n := f\chi_n$ for $n = 1, 2, \cdots$. Then $f_n \in C_c^2(E)$, $f_n \to f$, $\frac{\partial f_n}{\partial s} \to \frac{\partial f}{\partial s}$, $\frac{\partial f_n}{\partial x} \to \frac{\partial f}{\partial x}$ and $\frac{\partial^2 f_n}{\partial x^2} \to \frac{\partial f^2}{\partial x^2}$ uniformly on compacts as $n \to \infty$, and ($\|\cdot\|_\infty$ denotes the L^∞-norm)

$$\|f_n\|_\infty + \left\|\frac{\partial f_n}{\partial s}\right\|_\infty + \left\|\frac{\partial f_n}{\partial x}\right\|_\infty + \left\|\frac{\partial^2 f_n}{\partial x^2}\right\|_\infty \leq C < \infty \text{ for all } n \text{ (C is a constant).} \qquad (2.6)$$

From (Revuz and Yor, 1999, Prop.VII.1.6) and Thm.1, $M_t^{f_n}$ as defined in (2.5) by replacing f with f_n is a martingale.

Define $T_K = K \wedge \inf\{t : |X_{t-}^\psi| > K\}$ for $K = 1, 2, \cdots$, then T_K is a stopping time. Since X^ψ is conservative, $T_K \to \infty$ as $K \to \infty$. The stopped process $M_t^{f_n, K} := M_{t \wedge T_K}^{f_n}$ is also a martingale ((Revuz and Yor, 1999, Cor.II.3.6)). If we can prove $M_t^{f, K} := M_{t \wedge T_K}^f$ is a martingale, or equivalently, $\lim_{n\to\infty} \mathbb{E}[M_t^{f_n, K}|\mathcal{F}_s] = \mathbb{E}[M_t^{f, K}|\mathcal{F}_s]$ for $s \leq t$, then M_t^f is a local martingale. From (2.6) we can apply dominated convergence theorem to show the convergence for the first five terms in (2.5). For the last term, note that for all $s \leq T_K$, $|X_{s-}^\psi| \leq K$, and for all n and $(s, x) \in E$, $|f_n(s, x + y) - f_n(s, x) - \mathbf{1}_{\{|y|\leq 1\}}y\frac{\partial f_n}{\partial x}(s, x)| \leq C_1(y^2 \wedge 1)$ for some constant C_1. It is also not hard to show for all $|x| \leq K$, $\int_{y\neq 0}(y^2 \wedge 1)p(\tau, x, x + y)dy \leq C_2(\tau \wedge 1)$ for some constant C_2. This shows $\int_{[0,t\wedge T_K]\times\{y\neq 0\}} \left| f_n(s, X_{s-}^\psi + y) - f_n(s, X_{s-}^\psi) - \frac{\partial f_n}{\partial s}(s, X_{s-}^\psi) \right| \pi(s, X_{s-}^\psi, y)dsdy \leq C_1 C_2 \int_0^K \int_{(0,\infty)}(\tau \wedge 1)\nu(s, d\tau)ds < \infty$ for all n, which implies convergence by the dominated convergence theorem.

Step 2. To show X^ψ is a special semimartingale, from (Jacod and Shiryaev, 2003, Prop.II.2.29), it suffices to show for each T_K, $\mathbb{E}[\int_0^{T_K} \int_{|y|\neq 0}(y^2 \wedge |y|)\pi(s, X_{s-}^\psi, y)dyds] < \infty$. Note that for all $|x| \leq K$, $\int_{y\neq 0}(y^2 \wedge |y|)p(\tau, x, x+y)dy \leq C_3(\tau \wedge 1)$ for some constant C_3. Hence, $\int_0^{T_K} \int_{(0,\infty)} \int_{y\neq 0}(y^2 \wedge |y|)p(s, X_{s-}^\psi, X_{s-}^\psi + y)dy\nu(s, d\tau)ds \leq C_3 \int_0^K \int_{(0,\infty)}(\tau \wedge 1)\nu(s, d\tau)ds < \infty$ a.s. This shows the finiteness of the expectation. The sample path decomposition is a result of (Jacod and Shiryaev, 2003, Prop.II.2.29,Cor.II.2.38). $\square$

3   Option Pricing Under ASubOU-Based Commodity Models

Based on ASubOU processes, we develop a new class of commodity derivative models. Under the risk-neutral measure chosen by the market, the spot price S_t follows (1.1) with X^ϕ replaced by X^ψ, and $G(t) = \mathbb{E}[e^{X_t^\psi}]$. In order to obtain $G(t)$ and derivative prices, we study the transition operator $\mathcal{P}_{s,t}^\psi$ $(0 \le s < t < \infty)$ defined as

$$\mathcal{P}_{s,t}^\psi f(x) := \mathbb{E}[f(X_t^\psi)|X_s^\psi = x] = \int_{[0,\infty)} \mathcal{P}_u f(x) q_{s,t}(du). \tag{3.7}$$

In (3.7), $\mathcal{P}_{s,t}^\psi f(x)$ is well-defined and finite for a given x provided $\mathbb{E}[|f(X_t^\psi)||X_s^\psi = x] < \infty$. This is true in particular for measurable bounded functions. The equality comes from the sample path construction of X^ψ, where $(\mathcal{P}_u)_{u\ge 0}$ is the OU diffusion transition semigroup and $q_{s,t}$ is the transition probability measure for T from s to t. Since not every financial payoff is bounded, we have to go beyond this space and instead we consider $L^2(\mathbb{R}, m)$ payoffs, where the measure $m(dx) = m(x)dx$, and $m(x) = \sqrt{\frac{\kappa}{\pi\sigma^2}} \exp\{-\frac{\kappa(\theta-x)^2}{\sigma^2}\}$ is the OU diffusion stationary density. Given $f \in L^2(\mathbb{R}, m)$, we show that under some further conditions, $\mathcal{P}_{s,t}^\psi f(x)$ is well-defined and finite for all x and can be calculated by the following eigenfunction expansion which converges uniformly on compacts in x:

$$\mathcal{P}_{s,t}^\psi f(x) = \sum_{n=0}^\infty e^{-\int_s^t \psi(\kappa n, u)du} f_n \varphi_n(x), \ \ f_n = \int_\mathbb{R} f(x)\varphi_n(x)m(dx), \tag{3.8}$$

where $\varphi_n(x) = \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\right)$, and $H_n(x)$ is the Hermite polynomial of order n (e.g.Lebedev (1972)). Each $\varphi_n(x)$ is called an eigenfunction since $\mathcal{P}_t \varphi_n(x) = e^{-\kappa n t}\varphi_n(x)$, $\mathcal{P}_{s,t}^\psi \varphi_n(x)$ $= e^{-\int_s^t \psi(\kappa n, u)du}\varphi_n(x)$.

Theorem 3. (i) *For $f \in L^2(\mathbb{R}, m)$, if $\sum_{n=1}^\infty |f_n| n^{-\frac{1}{4}} < \infty$ then (3.8) is valid.*
(ii) If $\sum_{n=1}^\infty e^{-\int_s^t \psi(\kappa n,u)du} n^{-1/4} < \infty$, then (3.8) is valid for any $f \in L^2(\mathbb{R}, m)$.
Under both (i) and (ii), $\mathcal{P}_{s,t}^\psi f \in L^2(\mathbb{R}, m)$ and $\|\mathcal{P}_{s,t}^\psi f\| \le \|f\|$, where $\|\cdot\|$ denotes the L^2-norm.

Proof. The proof parallels (Li and Linetsky, 2012, Thm.2.23). For any $f \in L^2(\mathbb{R}, m)$, $\mathcal{P}_u f(x) = \sum_{n=0}^\infty e^{-\kappa n u} f_n \varphi_n(x)$, which converges uniformly on compacts in x. From (3.7), $\mathcal{P}_{s,t}^\psi f(x) = \int_{[0,\infty)} \sum_{n=0}^\infty e^{-\kappa n u} f_n \varphi_n(x) q_{s,t}(du) = \sum_{n=0}^\infty \int_{[0,\infty)} e^{-\kappa n u} q_{s,t}(du) f_n \varphi_n(x)$, which gives us (3.8) provided we can justify the interchange of summation and integration in the second equality. This, as well as the uniform convergence on compacts of (3.8) can be justified under the conditions in (i) and (ii) using similar arguments as in (Li and Linetsky, 2012, Thm.2.23). We omit the details here for the sake of brevity. We note that for each $u > 0$, $\|\mathcal{P}_u f\| \le \|f\|$, and from (3.7): $\int_\mathbb{R} \left(\mathcal{P}_{s,t}^\psi f(x)\right)^2 m(dx) \le \int_\mathbb{R} \int_{[0,\infty)} (\mathcal{P}_u f(x))^2 q_{s,t}(du)m(dx) = \int_{[0,\infty)} \|\mathcal{P}_u f\| q_{s,t}(du) \le \|f\|$. This shows the last statement. $\square$

The condition in Thm.3 (ii) is mild and satisfied for a wide class of additive subordinators of interest in finance. In particular, if the drift function $\gamma(\cdot)$ is not identically zero, then it is verified. Using Thm.3, pricing commodity derivatives under the ASubOU model is just as easy as the SubOU case. All the formulas in Li and Linetsky (2012) remain valid by substituting the Laplace exponent ϕ by its time-dependent counterpart ψ. This solves the problem for pricing futures and European futures options. For Bermudan futures options, Li and Linetsky (2013) calculated the continuation value at each time by an eigenfunction expansion, with the coefficients determined recursively. To

extend to the additive case, the formulas in (Li and Linetsky, 2013, Thm.3.2) hold true by replacing ϕ with ψ. Hence, pricing Bermudan options is also tractable under the ASubOU model. Li and Linetsky (2013) shows that American futures options under the SubOU model can be efficiently computed using Richardson extrapolation from Bermudan prices as the convergence pattern from Bermudan to American is linear and monotone. This pattern is again observed in the ASubOU model which suggests that extrapolation can also be applied here (see Figure 1 for an illustration). These results are quite remarkable, since Bermudan and American option pricing remains tractable even though the ASubOU process is a complicated time-inhomogeneous process with state- and time-dependent jumps.
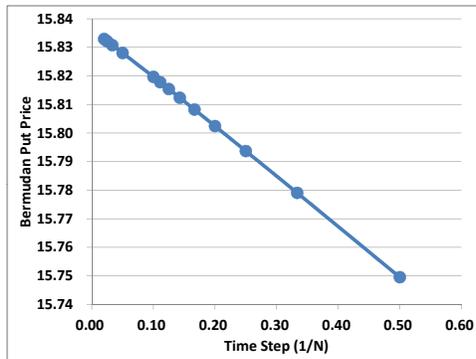

Figure 1: 1 year put option with the underlying futures contract maturing after 1.04 years. $F(0, 1.04) = 100$ and strike $= 105$. N is the number of Bermudan exercise dates with $N = 2, 3, 4, \cdots, 10, 20, 30, 40, 50$. $\theta = -1$, $\kappa = 0.2$, $\sigma = 0.35$, $x_0 = 0.0$. The IG-Sato subordinator (see section 4) is used with parameters $\gamma = 0.4$, $C = 0.48$, $\eta = 0.9$, $\rho = 0.8$. The linear regression $R^2 \geq 99.9\%$.

4 Examples of Additive Subordinators

We discuss two approaches to construct additive subordinators from known Lévy subordinators, which are abundant. This provides us with plenty of additive subordinators to use in practice. To simplify our exposition, we consider the following three parameter family of Lévy measures which are widely used in finance: $\nu(ds) = Cs^{-\alpha-1}e^{-\eta s}ds$, with $C > 0$, $\alpha < 1$, and $\eta \geq 0$. When $\alpha \in (0, 1)$, the Lévy measures correspond to the so-called *temperate stable subordinators*. In particular, if $\alpha = 1/2$ we have the *Inverse Gaussian (IG)* process. For $\alpha < 0$, the Lévy measures correspond to *compound Poisson processes*. Lastly, the limiting case for which $\alpha \to 0$, corresponds to the *gamma process*. The Laplace exponent for this family of subordinators with drift, $\gamma \geq 0$, is given by,

$$\phi(\lambda) = \begin{cases} \gamma\lambda - C\Gamma(-\alpha)\left((\lambda + \eta)^\alpha - \eta^\alpha\right), & \alpha \neq 0 \\ \gamma\lambda + C\ln(1 + \lambda/\eta), & \alpha = 0 \end{cases}. \tag{4.9}$$

Approach 1. The simplest approach is to make the parameters of some Lévy subordinator time dependent. For example, consider a time-dependent three-parameter family of Lévy measures. In this case, the Laplace exponent $\psi(\lambda, t)$ has the same functional form as $\phi(\lambda)$ of (4.9), where for all $t \geq 0$, $C(t) > 0$, $\alpha(t) < 1$, and $\gamma(t), \eta(t) \geq 0$, are functions of time. Such a specification is parameterized by four deterministic functions. However, if we have to calibrate to, say, a volatility surface on a time interval $[0, T]$ with N maturities, then one can reduce the number of degrees of

freedom by simply considering a piece-wise constant choice for these functions that are constant between maturities (this yields a characterization with $4 \times N$ parameters). In this case, one can easily show that $e^{-\int_0^t \psi(\lambda, u) du} = \exp\{-\sum_{i=0}^{k-1} \phi_i(\lambda)(t_{i+1} - t_i) - \phi_k(\lambda)(t - t_k)\}$, where k is the largest integer such that $t_k \leq t$ (here t_k are the nodes – maturity dates at which the parameters change, and $\phi_i(\lambda)$ are Laplace exponents with constant parameters on each of these time intervals from t_i to t_{i+1}). By appropriately choosing parameters, excellent calibration results are essentially assured to implied volatility across multiple maturities. However, the downside is that there may be too many parameters.

Approach 2. A much more parsimonious alternative is given by the so-called *Sato process with increasing paths.* Let $\rho > 0$ denote a self-similarity index. A process $(T_t)_{t \geq 0}$ is called a ρ-Sato process if it is additive (i.e., with independent increments), and is ρ-self-similar (i.e., $(T_{ct})_{t \geq 0} \overset{(law)}{=} (c^\rho T_t)_{t \geq 0}$, for all $c > 0$). From (Sato, 1999, Thm.16.1), a r.v. Z is self-decomposable if and only for every $\rho > 0$, there exists a ρ-Sato process $(T_t)_{t \geq 0}$ such that $T_1 \overset{(law)}{=} Z$. In particular, if the self-decomposable law of Z has support on $\mathbb{R}_+$, then the Sato process has increasing paths. From (Sato, 1999, Cor.15.11), the Laplace transform of Z, satisfies $\mathbb{E}[e^{-\lambda Z}] = e^{-\phi(\lambda)}$, where $\phi(\lambda) := \gamma\lambda + \int_0^\infty (1 - e^{-\lambda s}) \frac{h(s)}{s} ds$, for all $\lambda > 0$ and $\gamma \geq 0$. The function $h(s)$ is positive and decreasing on $(0, \infty)$. From (Carr et al., 2003, Thm.1), the ρ-Sato process T_t with increasing paths is an additive subordinator with drift $\gamma(t) = \gamma\rho t^{\rho-1}$ and Lévy measure $\nu(t, ds) = -\frac{\rho h'(st^{-\rho})}{t^{\rho+1}} ds$ such that $\mathbb{E}[e^{-\lambda T_t}] = e^{-\int_0^t \psi(\lambda, u) du} = e^{-\phi(\lambda t^\rho)}$ for all $\rho, \lambda > 0$. Consequently, one can easily construct Sato-type additive subordinators from Lévy subordinators with self-decomposable distributions, e.g., tempered stable subordinators. For instance, the Laplace exponent for the IG-Sato and Gamma-Sato additive subordinators can be readily obtained from the Laplace exponent $\phi(\lambda)$ of Eq. (4.9).

5 Calibration Examples to Multiple Maturities

Our goal is to find a more parsimonious alternative than the TDSVSubOU model used in Li and Linetsky (2012) for calibration to multiple maturities, and at the same time more tractable and faster for option pricing. To be parsimonious we consider the IG-Sato subordinator, which only introduces one additional parameter compared to a SubOU model using an IG subordinator. To compare with the TDSVSubOU model proposed in Li and Linetsky (2012), we redo calibrations for crude oil and zinc as considered in Li and Linetsky (2012) using the same data. Li and Linetsky (2012) considered 4 maturities (approximately 0.5, 1, 1.5 and 2 years). Following Carr et al. (2007), moneyness (defined as strike/initial futures price) ranges between 0.6 and 1.4. As in Li and Linetsky (2012), we minimize the sum of squared differences between model and market implied volatilities. The goodness of fit is evaluated using average percentage error (APE) as in Carr et al. (2007), which is defined as the average absolute error divided by the average option price (all options used are OTM except the ATM ones). It is pointed out in Carr et al. (2007) that market practice regards a particular model as having failed if APE exceeds 5%. We set $x_0 = 0$ in our calibration as θ can be changed $\theta - x_0$ without affecting the European option prices. The calibration results are summarized in Table 1. Under the TDSVSubOU model, Li and Linetsky (2012) set the deterministic activity rate $a(\cdot)$ to be piecewise constant (PC) where $a(\cdot)$ is a constant for each maturity, and used IG as the Lévy subordinator. The SubOU model fails to calibrate crude oil and both the TDSVSubOU and the ASubOU model outperform the SubOU model substantially. It is not surprising the TDSVSubOU model has smaller APE than the ASubOU model, as additional parameters provide more flexibility in calibration. However, the ASubOU model is much more parsimonious and its APE is small enough to be considered as very good according to market practice. Pricing

	No. Parameters	Crude Oil APE	Zinc APE
SubOU (IG)	6	5.52%	2.19%
TDSVSubOU (PC-CIR-IG)	14	0.72%	0.45%
ASubOU (IG-Sato)	7	1.39%	0.58%

Table 1: Comparison of SubOU, TDSVSubOU and ASubOU models

European options is also much faster under the ASubOU model than the TDSVSubOU model. Under the TDSVSubOU model, the European option price is expressed as an integral with the integrand expressed as an eigenfunction expansion (c.f.(Li and Linetsky, 2012, Thm.4.5)). To evaluate it numerically one has to discretize the integral (usually over 50 intervals are needed to get good accuracy using Simpson's rule), which results in computing more than 100 eigenfunction expansions. In contrast under the ASubOU model only one eigenfunction expansion needs to be computed. This, together with the significant reduction in the number of parameters, makes calibration of the ASubOU model much faster, a key aspect in market practice. Furthermore, the ASubOU model is completely tractable for pricing Bermudan and American options while it is difficult to do so under the TDSVSubOU model.

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