Equivalent Measure Changes for Subordinate Diffusions

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Abstract

A subordinate diffusion is a Markov jump-diffusion or pure jump process obtained by time changing a diffusion process with an independent Lévy or additive subordinator. This class of processes has found many applications in finance. In this paper, we derive sufficient conditions for equivalent measure changes for subordinate diffusions that are easy to check in typical financial applications.

Key Words: equivalent measure changes, Bochner's subordination, additive subordination, jump processes.

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1 Introduction

Bochner's subordination, first introduced by S. Bochner [5, 6], is a time-honored method in probability that constructs new time-homogeneous Markov processes by time changing existing time-homogeneous Markov processes with independent Lévy subordinators (i.e., nonnegative and nondecreasing Lévy processes). In particular, applying Bochner's subordination to diffusion processes provides a rich family of jump-diffusions and pure jump processes whose jumps are generally

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state-dependent which exhibit a variety of interesting behavior. For example, if the background diffusion is a mean-reverting process, then the time-changed process exhibits mean-reverting jumps. In addition, jumps could be of finite or infinite activity and finite or infinite variation. This class of processes is called Bochner/Lévy subordinate diffusions and they have found many applications in finance. Many popular Lévy processes used for financial modelling can be represented as a subordinate Brownian motion. Examples include Variance Gamma [20], Normal Inverse Gaussian [3], CGMY [8], generalized Hyperbolic [11] and Meixner [28]. Applications of Bochner's subordination to more general diffusions include [4] for equity, [23] and [19] for credit-equity derivatives, [17] for commodity derivatives, [7] and [18] for interest rates, and [22] for credit default.

Recently, in [16], the authors of the present paper propose additive subordination as a generalization of Bochner's subordination to develop time-inhomogeneous Markov processes from existing time-homogeneous Markov processes. The idea is to use additive subordinators (nonnegative and nondecreasing additive processes; see [16]), which are independent of the background process as the time change. Unlike Lévy subordinators, additive subordinators do not have stationary increments, hence the time-changed process is time-inhomogeneous. The application of additive subordination to time-homogeneous diffusions develops a rich class of jump-diffusions and pure jump processes with time-dependent and state-dependent local characteristics. We call such processes as additive subordinate diffusions. The financial applications of these processes have already been explored in [16], where it is shown that the time-dependent characteristics allow one to achieve excellent calibration results to the implied volatility surface in commodity markets.

The purpose of this paper is to develop sufficient conditions for equivalent measure changes for additive subordinate diffusions that are easy-to-verify in typical examples in finance (the classical Lévy subordinate diffusions are special cases under this general framework). To our best knowledge, such results are still lacking in the literature. The problem is important for financial applications, as the standard no-arbitrage theory states that the pricing probability measure for financial derivatives valuation is equivalent to the probability measure under the real-world (see [10]). So it is natural to develop a class of equivalent measure changes for additive subordinate diffusions to transform the financial model based on these processes from one measure to the other.

There already exist various results for equivalent measure changes for different types of Markov jump processes. See, e.g., [26] for sufficient and necessary conditions for Lévy processes, and [9] for sufficient conditions for a general class of Markov processes with jumps. These conditions can be directly checked for some popular jump-diffusions in finance, including, e.g., the double-exponential jump-diffusion [15], the basic affine jump-diffusion [9] and more generally, Lévy processes and affine jump-diffusions with explicit jump intensity.

To develop equivalent measure changes for additive subordinate diffusions, we first establish the well-posedness of the martingale problem (Theorem 3.1) for this class of processes by proving the well-posedness of a homogeneous Cauchy problem and analyzing the core of the induced evolution semigroup associated with the space-time process. Then, based on the semimartingale characterization of additive subordinate diffusions developed in [16] and equivalent measure change results for general semimartingales developed in [13], we derive two conditions under which an additive subordinate diffusion is equivalent to another additive subordinate diffusion (Theorem 3.2). The first condition requires the quadratic variation of the continuous local martingale part to remain unchanged. The second condition states that the Hellinger distance between the jump intensities under two measures is bounded on compacts. When the background diffusion is a Brownian motion and the time change is a Lévy subordinator, our conditions coincide with those conditions for equivalent measure changes for Lévy processes (see [26]).

Unlike jump processes with explicit jump intensity for which the verification of the Hellinger condition is straightforward, an additional complexity arises in our problem due to the generally unknown jump intensity of additive subordinate diffusions, which is given by an integral involving the transition probability density of the background diffusion and the Lévy measure of the subordinator (see Eq.(2.5)). While the Lévy measure is given explicitly in applications, the diffusion transition density is in general unknown. Even if it is known, the integral defining the jump intensity may not be available in closed-form. We tackle this problem with the help of the short time expansion for diffusion transition densities derived in [1, 2].

The rest of the paper is organized as follows. In Section 2, we specify the class of additive subordinate diffusions we work with and construct them as the canonical process on the Skorokhod space. In Section 3, we establish the well-posedness of the martingale problem for additive subordinate diffusions, derive sufficient conditions for equivalent measure changes for them and show how to verify the Hellinger condition. Section 4 applies results in Section 3 to the additive subordinate CIR process, which is used for commodity modelling in [16]. In this case, the conditions for equivalent measure changes reduce to explicit restrictions on the model parameters.

2 Additive Subordinate Diffusions

We start with a diffusion $(D_t)_{t\geq 0}$ living on $I=(l,r)$ with $-\infty\leq l<r\leq\infty$. For simplicity, in this paper we restrict our discussions to one-dimensional additive subordinate diffusions. However, our arguments potentially can be applied to deal with multidimensional ones.

We assume D is regular, i.e., for any $x,y\in I$, it is possible to reach y in finite time starting from x. The drift and diffusion coefficient for D are denoted by $\mu(x)$ and $\sigma(x)$, respectively. We assume the following.

Assumption 2.1. *$\mu(x)$ and $\sigma(x)$ are continuous on I with $\sigma(x)>0$ for $x\in I$. l and r are inaccessible endpoints for the diffusion.*

We refer readers to e.g., [14], Definition 7.3.1, for the classification of boundary behavior for one-dimensional diffusions. Under this assumption, l and r can only be natural or entrance endpoints.

Let $p(t,x,y)$ be the transition density of D (its existence in shown in [21]), and $\mathcal{P}_t^D$ is the transition operator, i.e., for measurable and bounded function f on I,

$$\mathcal{P}_t^D f(x)=\int_I f(y)p(t,x,y)dy.$$

From Theorem 7.2.2 in [14], the transition semigroup $(\mathcal{P}_t^D)_{t\geq 0}$ is a strongly continuous semigroup of contractions on $C_b(I)$ (continuous and bounded functions on I), and its infinitesimal generator $\mathcal{G}^D$ is given by

$$\mathcal{G}^D f(x)=\tfrac{1}{2}\sigma^2(x)f''(x)+\mu(x)f'(x), \tag{2.1}$$

with $\mathfrak{D}(\mathcal{G}^D)=\{f\in C_b(I):\frac{1}{2}\sigma^2(x)f''(x)+\mu(x)f'(x)\in C_b(I)\}$.

An additive subordinator is a nondecreasing additive process taking values in $[0,\infty)$. We consider an additive subordinator $(T_t)_{t\geq 0}$ with differential characteristics (see [16], Section 2). We denote the distribution of T_t-T_s by $q_{s,t}$ $(0\leq s\leq t<\infty)$. The Laplace transform of T_t-T_s is given by ([16], Proposition 2.1)

$$\int_{[0,\infty)} e^{-\lambda\tau} q_{s,t}(d\tau) = e^{-\int_s^t \psi(\lambda,u)du}, \ \psi(\lambda,u) = \lambda\gamma(u) + \int_{(0,\infty)} \left(1 - e^{-\lambda\tau}\right) \nu(u,d\tau), \ \lambda > 0.$$

Assumption 2.2. *We assume for all $t \geq 0$,*

$$\gamma(t) \geq 0, \ \int_{(0,\infty)} (\tau \wedge 1)\nu(t,d\tau) < \infty,$$

and $\nu(t,\cdot)$ is not identically zero. Define $\nu_F(t,A) := \int_A (\tau \wedge 1)\nu(t,d\tau)$ for any Borel set $A \subseteq (0,\infty)$, then $\nu_F(t,\cdot)$ is a finite measure on $(0,\infty)$ for all $t \geq 0$. We assume $\gamma(t)$ is continuous in t, and $\nu_F(z,\cdot)$ converges weakly to $\nu_F(t,\cdot)$ as $z \to t$.

Remark 2.1. *A Lévy subordinator is an additive subordinator with stationary increments so that $\gamma(t)$ and $\nu(t,\cdot)$ do not depend on t.*

Now we construct a two-parameter family of operators by applying additive subordination to the diffusion transition semigroup. Define $\mathcal{P}_{s,t}f$ for $f \in C_b(I)$ as

$$\mathcal{P}_{s,t}f(x) := \int_{[0,\infty)} \mathcal{P}_\tau^D f(x) q_{s,t}(d\tau). \tag{2.2}$$

From Theorem 3.1 in [16], $(\mathcal{P}_{s,t})_{0 \leq s \leq t}$ is a strongly continuous backward propagator as well as propagator of contractions on $C_b(I)$ (for the definition of these concepts, see [16], Section 3). The next proposition obtains continuity properties for $\mathcal{P}_{s,t}f$.

Proposition 2.1. *(1) For every $f \in C_b(I)$, $\mathcal{P}_{s,t}f(x)$ is jointly continuous in s,t,x. (2) Suppose $s_n \to s$, $t_n \to t$ and $f_n \to f$ in $C_b(I)$. Then $\mathcal{P}_{s_n,t_n}f_n \to \mathcal{P}_{s,t}f$ in $C_b(I)$.*

Proof of Proposition 2.1. We first point out that for the diffusions we consider, it is shown in [21] that $\mathcal{P}_\tau^D f(x)$ is jointly continuous in τ and x for $f \in C_b(I)$.

(1) We want to show that for given (s,t,x), as $s_n \to s$, $t_n \to t$ $(s_n \leq t_n)$ and $x_n \to x$,

$$\lim_{n\to\infty} \mathcal{P}_{s_n,t_n}f(x_n) = \mathcal{P}_{s,t}f(x).$$

Define $g_n(\tau) := \mathcal{P}_\tau f(x_n)$ and $g(\tau) := \mathcal{P}_\tau f(x)$. Both g_n and g are continuous in τ, and they are bounded by the maximum norm of f. For every τ, $g_n(\tau) \to g(\tau)$. We write

$$\begin{aligned}\mathcal{P}_{s_n,t_n} f(x_n) - \mathcal{P}_{s,t} f(x) &= \int_{[0,\infty)} g_n(\tau) q_{s_n,t_n}(d\tau) - \int_{[0,\infty)} g(\tau) q_{s_n,t_n}(d\tau) \\ &\quad + \int_{[0,\infty)} g(\tau) q_{s_n,t_n}(d\tau) - \int_{[0,\infty)} g(\tau) q_{s,t}(d\tau).\end{aligned}$$

For the first difference, using the dominated convergence theorem for varying measures ([29], Theorem 2.4), it converges to zero. The second difference also converges to zero as q_{s_n,t_n} converges to $q_{s,t}$ weakly, which is implied by the convergence of the Laplace transform. This shows the joint continuity.

(2) We use $\|\cdot\|$ for the maximum norm. We have

$$\|\mathcal{P}_{s_n,t_n} f_n - \mathcal{P}_{s,t} f\| = \|\mathcal{P}_{s_n,t_n} f_n - \mathcal{P}_{s_n,t_n} f\| + \|\mathcal{P}_{s_n,t_n} f - \mathcal{P}_{s,t} f\|.$$

The first part is bounded by $\|f_n - f\|$ so it converges to zero. The second part also converges to zero due to the strong continuity of $(\mathcal{P}_{s,t})_{0\le s\le t}$. This proves the claim. □

Proposition 2.1 (1) together with Remark 2.5 in [30] imply that $(\mathcal{P}_{s,t})_{0\le s\le t}$ is a Feller propagator in the sense of Definition 2.4 in [30]. From Theorem 2.9 of [30], one can construct a strong time-inhomogeneous Markov process $(\Omega, \mathcal{F}, X_t, \mathcal{F}_{s,t}, P_{s,x})$ associated with $(\mathcal{P}_{s,t})_{0\le s\le t}$. Here

- Ω is the Skorokhod space that consists of càdlàg functions $\omega : [0,\infty) \mapsto I$.
- X is the canonical process on Ω, i.e., $X_t(\omega) = \omega(t)$.
- $\mathcal{F}_{s,t} := \mathcal{F}^0_{s,t+} = \bigcap_{\tau>t} \mathcal{F}^0_{s,\tau}$, where $\mathcal{F}^0_{s,t} := \sigma(X_u : s \le u \le t)$ is the double filtration generated by X (see [30], Definition 2.14). $\mathcal{F} = \bigvee_{t\ge 0} \mathcal{F}^0_{0,t}$ is the smallest sigma-field that contains all $\mathcal{F}^0_{0,t}$.
- $\{P_{s,x}\}$ is a family of probability measures on $(\Omega, \mathcal{F})$ such that $P_{s,x}(X_s = x) = 1$ and $E_{s,x}[f(X_t)] = \mathcal{P}_{s,t} f(x)$.

We now give the semimartingale characterization for the additive subordinate diffusion X. For ease of discussions, we extend $p(\tau, x, y)$ from $y \in I$ to $y \in \mathbb{R}$ by defining $p(\tau, x, y) = 0$ for $y \notin I$.

Define

$$a(t,x) := \sqrt{\gamma(t)}\sigma(x), \tag{2.3}$$

$$b(t,x) := \gamma(t)\mu(x) + \int_{(0,\infty)} \left(\int_{\{|y|\le 1\}} y p(\tau, x, x+y) dy \right) \nu(t, d\tau), \tag{2.4}$$

$$\Pi(t,x,dy) := \pi(t,x,y)dy, \quad \pi(t,x,y) := \int_{(0,\infty)} p(\tau,x,x+y)\nu(t,d\tau) \quad \text{for } y \neq 0. \tag{2.5}$$

Remark 2.2. *It is shown in [16] that $\Pi(t,x,dy)$ is a Lévy-type measure, i.e.,*

$$\int_{y\neq 0} (1 \wedge y^2)\pi(t,x,y)dy < \infty. \tag{2.6}$$

For each (t,x), $\int_{y\neq 0} \pi(t,x,y)dy < \infty$ if and only if $\int_{(0,\infty)} \nu(t,d\tau) < \infty$. This is because

$$\begin{aligned}\int_{y\neq 0} \pi(t,x,y)dy &= \int_{y\neq 0}\int_{(0,\infty)} p(\tau,x,x+y)\nu(t,d\tau)dy \\ &= \int_{(0,\infty)}\int_{y\neq 0} p(\tau,x,x+y)dy\nu(t,d\tau) = \int_{(0,\infty)} \nu(t,d\tau).\end{aligned}$$

The interchange of integration order is justified by Tonelli's Theorem.

Theorem 4.2 in [16] shows that for every $s \ge 0$ and $x \in I$, $(X_t)_{t\ge s}$ is a semimartingale w.r.t. $(\mathcal{F}_{s,t})_{t\ge s}$ under $P_{s,x}$, with characteristics triplet (B, C, ν) w.r.t. the truncation function $h(x) = x1_{\{|x|\le 1\}}$, where

$$B_t(\omega) := \int_0^t b(s, X_{s-}(\omega))ds, \tag{2.7}$$

$$C_t(\omega) := \int_0^t a^2(s, X_{s-}(\omega))ds, \tag{2.8}$$

$$J(\omega, dt, dy) := \pi(t, X_{t-}(\omega), y)dydt. \tag{2.9}$$

We refer readers to [13], Section II.2 for the definition of semimartingale characteristics. Here B_t is the predictable finite variation part which gives the drift of X. C_t is the quadratic variation process of the continuous local martingale part of X. J is the compensator of the random jump measure of X, which is absolutely continuous and gives the intensity of jumps into another state in I.

3 Equivalent Measure Changes

We call $(\mu(x), \sigma(x), \gamma(t), \nu(t, \cdot))$ the generating tuple for the additive subordinate diffusion X under the family of probability measures $\{P_{s,x}\}$. Now given another tuple $(\bar{\mu}(x), \bar{\sigma}(x), \bar{\gamma}(t), \bar{\nu}(t, \cdot))$ which satisfies Assumption 2.1 and 2.2, suppose that there exists a unique family of probability measures $\{\bar{P}_{s,x}\}$ defined on $(\Omega, \mathcal{F})$, such that X is an additive subordinate diffusion with the given tuple as the generating tuple. We are interested in developing conditions under which, $\bar{P}_{s,x}|_{\mathcal{F}_{s,t}} \sim P_{s,x}|_{\mathcal{F}_{s,t}}$ for every $t \geq s$, i.e., $\bar{P}_{s,x}$ and $P_{s,x}$ are equivalent.

3.1 Well-Posedness of the Martingale Problem

For a given tuple, we first need to address the issue whether there exists a unique family of probability measures for X to be an additive subordinate diffusion. This question is equivalent to the well-posedness of the following martingale problem considered in [13], Section III.2 (we make some modifications to adapt the problem to our setting for time-inhomogeneous Markov processes). Given the Skorokhod space Ω, the canonical process X, the filtration $(\mathcal{F}_{s,t})_{0\leq s\leq t}$ and the sigma-field $\mathcal{F}$ defined in Section 2, as well as a generic tuple $(\mu(x), \sigma(x), \gamma(t), \nu(t, \cdot))$ which satisfies Assumption 2.1 and 2.2, define $a(t, x)$, $b(t, x)$, $\Pi(t, x, dy)$, $B_t(\omega)$, $C_t(\omega)$ and $J(\omega, dt, dy)$ by (2.3), (2.4), (2.5), (2.7), (2.8) and (2.9), respectively. A family of probability measures $\{P_{s,x}\}$ is a solution to the martingale problem associated with (B, C, J) if under $P_{s,x}$, $P_{s,x}(X_s = x) = 1$ and $(X_t)_{t\geq s}$ is a semimartingale w.r.t. $(\mathcal{F}_{s,t})_{t\geq s}$ with characteristics triplet (B, C, J) w.r.t. the truncation function $h(x) = x1_{\{|x|\leq 1\}}$. If the solution exists and is also unique, then the martingale problem is called well-posed.

The existence of a solution to our martingale problem is quite obvious as it can be constructed by additive subordination. In details, there exist a unique diffusion transition semigroup $(\mathcal{P}_t)_{t\geq 0}$ associated with $(\mu(x), \sigma(x))$ ([14], Corollary 7.2.2) and a unique additive subordinator associated with $(\gamma(t), \nu(t, \cdot))_{t\geq 0}$. We can first construct a Feller propagator $(\mathcal{P}_{s,t})_{0\leq s\leq t}$ by applying additive subordination to $(\mathcal{P}_t)_{t\geq 0}$ (see (2.2)) and then construct a family of probability measures on the Skorokhod space associated with $(\mathcal{P}_{s,t})_{0\leq s\leq t}$. Our next proposition shows that the solution is also unique.

Theorem 3.1. *The solution to the martingale problem for X associated with (B, C, J) is unique.*

To prove this theorem, we need the following lemma. Recall that $\mathcal{G}^D$ is the diffusion generator (see (2.1)).

Lemma 3.1. *For any $f \in \mathfrak{D}(\mathcal{G}^D)$, $\mathcal{P}_{s,t}f \in \mathfrak{D}(\mathcal{G}^D)$, that is, $\mathfrak{D}(\mathcal{G}^D)$ is invariant under $\mathcal{P}_{s,t}$.*

Proof of Lemma 3.1. The proof can be done similarly as the proof in [26], p.215, lines 5-10, by replacing the transition probability measure of Lévy subordinators by that of additive subordinators. The details are omitted here. □

Now we prove well-posedness of the martingale problem. The key step is to establish the well-posedness of a homogeneous Cauchy problem (see (3.2)).

Proof of Theorem 3.1. Consider the space-time process (t, X_t). The solution to the martingale problem for X associated with (B, C, J) is unique if and only if the solution to the corresponding martingale problem for the space-time process is unique. Let $P_{s,x}$ be a solution to the martingale problem. From Theorem II.2.42 (c) in [13], for any bounded $f(t,x)$ that is once-continuously differentiable in t and twice continuously differentiable in x, under $P_{s,x}$,

$$\begin{aligned} M^f_{s,t,x} &:= f(t,X_t) - f(s,x) \\ &- \int_s^t \left[\frac{\partial f}{\partial s}(s,X_{s-}) + \frac{1}{2}a^2(s,X_{s-})\frac{\partial^2 f}{\partial x^2}(s,X_{s-}) + b(s,X_{s-})\frac{\partial f}{\partial x}(s,X_{s-})\right. \\ &+ \left.\int_{y\neq 0}\left(f(s,X_{s-}+y) - f(s,X_{s-}) - 1_{\{|y|\leq 1\}}y\frac{\partial f}{\partial x}(s,X_{s-})\right)\Pi(s,X_{s-},dy)\right] ds \end{aligned} \tag{3.1}$$

is a local martingale.

Recall the expression for the diffusion generator $\mathcal{G}^D$ and its domain (see (2.1)). Note that $\mathfrak{D}(\mathcal{G}^D)$ is dense in $C_b(I)$. Define a family of linear operators $(\mathcal{G}_t)_{t\geq 0}$ on $\mathfrak{D}(\mathcal{G}^D)$ by

$$\mathcal{G}_t f(x) = \gamma(t)\mathcal{G}^D f + \int_{(0,\infty)} (\mathcal{P}^D_\tau f(x) - f(x))\nu(t,d\tau),$$

where $(\mathcal{P}^D_\tau)_{\tau\geq 0}$ is the diffusion transition semigroup. Using the arguments in the proof of Theorem 4.1 in [16], for $f \in \mathfrak{D}(\mathcal{G}^D)$, $\mathcal{G}_t f(x)$ can be expressed as

$$\mathcal{G}_t f(x) = \frac{1}{2}a^2(t,x)f''(x) + b(t,x)f'(x) + \int_{y\neq 0}\left(f(x+y) - f(x) - 1_{\{|y|\leq 1\}}yf'(x)\right)\Pi(t,x,dy).$$

We consider the homogeneous Cauchy problem (CP) on $\mathfrak{D}(\mathcal{G}^D)$.

$$\frac{d}{dt}u(t) = \mathcal{G}_t u(t),\ u(s) = f,\ t \geq s \geq 0. \tag{3.2}$$

The problem is called well-posed on $\mathfrak{D}(\mathcal{G}^D)$ if $\mathfrak{D}(\mathcal{G}^D)$ is dense in $C_b(I)$, and $\mathfrak{D}(\mathcal{G}^D) \subseteq \mathfrak{D}(\mathcal{G}_s)$ for all $s \geq 0$ such that for each $f \in \mathfrak{D}(\mathcal{G}^D)$, there is a unique solution $u(t; s, f)$ which is once continuously differentiable in t on $[s, \infty)$ with $u(t; s, f) \in \mathfrak{D}(\mathcal{G}^D)$ for all $t \geq s$, and if $s_n \to s$ and $f_n \to f$ in $C_b(I)$, then $\hat{u}(t; s_n, f_n) \to \hat{u}(t; s, f)$ in $C_b(I)$ uniformly for t on compacts. Here $\hat{u}(t; s, f) := u(t; s, f)$ if $t \geq s$ and $\hat{u}(t; s, f) := f$ for $t \leq s$ (see [27], Definition 2.1).

We want to show in our case the problem is indeed well-posed. From Theorem 3.1 in [16], the unique solution to (3.2) is given by $u(t; s, f) = \mathcal{P}_{s,t} f$ defined in (2.2). Under Assumption 2.2, it can be shown that for any $f \in \mathfrak{D}(\mathcal{G}^D)$, $\mathcal{G}_t$ is continuous in t, which implies that $u(t; s, f)$ is C^1 in t. Lemma 3.1 shows that $u(t; s, f) \in \mathfrak{D}(\mathcal{G}^D)$. Finally, we consider whether $\hat{u}(t; s_n, f_n) \to \hat{u}(t; s, f)$ uniformly on compacts for t. We only consider the case that $t \in K$ for a compact set K such that all s_n and s are outside K. The other cases can be proved with easy modifications. From Proposition 2.1 (2), we have $u(t; s, f)$ is jointly continuous in t, s, f, hence uniformly jointly continuous on any compact set of the form $K_t \times K_s \times K_f$, where K_t, K_s and K_f are compact sets for t, s and f, respectively. This implies that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $|t_1 - t_2| < \delta$, $|s_1 - s_2| < \delta$ and $\|f_1 - f_2\| < \delta$, $\|u(t_1; s_1, f_1) - u(t_2; s_2, f_2)\| < \varepsilon$. Now suppose the given $(t, s, f) \in K_t \times K_s \times K_f$. When n is sufficiently large, $s_n \in K_s$, $f_n \in K_f$ and $|s_n - s| < \delta$ and $\|f_n - f\| < \delta$. Therefore, for any $t \in K_t$, $\|u(t; s_n, f_n) - u(t; s, f)\| < \varepsilon$, which shows the convergence is uniform on K_t.

Now consider the induced evolution semigroup of $(\mathcal{P}_{s,t})_{0 \leq s \leq t}$ on the space $E = C_{00}([0, \infty), C_b(I))$, which consists of functions $f(t, x)$ such that $f(0, x) = 0$, $\lim_{t \to \infty} \sup_{x \in I} f(t, x) = 0$, and for each $t > 0$, $f(t, x) \in C_b(I)$. Its generator is denoted by $\mathcal{A}$. We have verified the condition of Proposition 4.1 in [27]. This proposition implies that $F = \{f \in C^1([0, \infty), C_b(I)) : f(t, x) \in \mathfrak{D}(\mathcal{G}^D)$ for each $t \geq 0, f, \frac{\partial f}{\partial t}, \mathcal{G}_t f \in E\}$ is a core of $\mathcal{A}$, and that $\mathcal{A} f(t, x) = \frac{\partial f}{\partial t}(t, x) + \mathcal{G}_t f(t, x)$ on F. Here C^1 means once continuously differentiable in t. If $f(t, x) \in F$, $f(t, x)$ is bounded, and it is once-continuously differentiable in t and twice continuously differentiable in x. Then we have for $f(t,x) \in F$,

$$M^f_{s,t,x} = f(t, X_t) - f(s,x) - \int_s^t \mathcal{A}f(s, X_{s-})ds,$$

is a local martingale under $P_{s,x}$. Note that $(M^f_{s,t,x})_{0\le s\le t}$ is bounded for each $t > 0$ (recall that $M^f_{s,t,x}$ is defined in (3.1)). Hence, by Theorem 51 in [24], Chapter 1, $M^f_{s,t,x}$ is a martingale under $P_{s,x}$. From this we can see that if $P_{s,x}$ is a solution to the martingale problem defined in this paper, it is also a solution to the martingale problem in the sense of [12]. Hence, we apply Theorem 4.1 and Corollary 4.3 in [12], Chapter 4, which show that the solution to the martingale problem for the space-time process (t, X_t) is unique. □

3.2 Sufficient Conditions for Equivalent Measure Changes

We are given two tuples $(\mu(x), \sigma(x), \gamma(t), \nu(t,\cdot))$ and $(\bar{\mu}(x), \bar{\sigma}(x), \bar{\gamma}(t), \bar{\nu}(t,\cdot))$, both of which satisfy Assumption 2.1 and 2.2. Let $P_{s,x}$ and $\bar{P}_{s,x}$ be the corresponding unique solution of the martingale problem (Theorem 3.1). Below we develop sufficient conditions for $P_{s,x}$ and $\bar{P}_{s,x}$ to be equivalent based on measure change results for general semimartingales.

Theorem 3.2. *Suppose the following conditions are satisfied,*

(i) $\gamma(s)\sigma(x)^2 = \bar{\gamma}(s)\bar{\sigma}(x)^2$ *for almost all* $x \in I$ *and* $s \ge 0$.

(ii) (Hellinger Condition) The integral

$$\int_{y\neq 0} \left(\sqrt{\bar{\pi}(s,x,y)} - \sqrt{\pi(s,x,y)}\right)^2 dy \text{ is bounded on } [0,t]\times K,$$

for any finite $t > 0$ *and any compact subset* K *of* I.

Then for any $s \ge 0$, $x \in I$, $\bar{P}_{s,x}|_{\mathcal{F}_{s,t}} \sim P_{s,x}|_{\mathcal{F}_{s,t}}$ *for every* $t \ge s$.

Proof of Theorem 3.2. : We will show that under our conditions (i)–(ii), conditions of Theorem III.5.34 of [13] are satisfied, which implies $\bar{P}_{s,x}|_{\mathcal{F}_{s,t}} \ll P_{s,x}|_{\mathcal{F}_{s,t}}$ for every $t \ge s$. Then reversing the role of $\bar{P}_{s,x}$ and $P_{s,x}$ and applying the same arguments give us their equivalence. Define

$$\beta_t(\omega) := \frac{\bar{\gamma}(t)\bar{\mu}(X_{t-}(\omega)) - \gamma(t)\mu(X_{t-}(\omega))}{\gamma(t)\sigma^2(X_{t-}(\omega))} 1_{\{\gamma(t)\sigma^2(X_{t-}(\omega))\neq 0\}},\ Y(\omega,t,y) := \frac{\bar{J}(\omega,dt,dy)}{J(\omega,dt,dy)}.$$

Condition (i) implies for all ω, the quadratic variations of the continuous local martingale parts are equal, i.e., $\bar{C} = C$. We also have $\bar{J} = Y \cdot J$ from the definition of Y. We next verify

$$\bar{B} = B + \gamma\sigma^2\beta \cdot t + h(x)(Y-1) * J. \tag{3.3}$$

Condition (ii) implies that,

$$\int_{y\neq 0} \left(\sqrt{\bar{\pi}(t,x,y)} - \sqrt{\pi(t,x,y)}\right)^2 dy < \infty, \tag{3.4}$$

By replacing the Lévy measure used in Remark 33.3, lines 17-19 of [26] by our state-dependent jump measure, we can show that (3.4) implies that

$$\int_{|y|\leq 1} |y| \cdot |\bar{\pi}(t,x,y) - \pi(t,x,y)| dy < \infty, \tag{3.5}$$

so $h(x)(Y-1) * J$ is finite (recall that $h(x) = x1_{\{|x|\leq 1\}}$). Under (3.5), we can apply the dominated convergence theorem, and obtain

$$\begin{aligned} &\int_{(0,\infty)} \int_{|y|\leq 1} y\bar{p}(\tau, x, x+y) dy \bar{\nu}(t, d\tau) - \int_{(0,\infty)} \int_{|y|\leq 1} y p(\tau, x, x+y) dy \nu(t, d\tau) \\ &= \int_{|y|\leq 1} y \Big[\bar{\pi}(t,x,y) - \pi(t,x,y)\Big] dy. \end{aligned} \tag{3.6}$$

From the expression in (2.4), (2.7), and (3.6), it is straightforward to verify (3.3) for all ω. Thus we have shown that [13], III.5.5 holds.

Next, let σ_{JS} and H be the predictable time and predictable process defined in [13], III.5.6 and III.5.7, respectively. Since $J(\omega, \{t\}, dx) = 0$, it is clear that $\sigma_{JS} = \infty$. Meanwhile, in our case, the process H becomes

$$H_t(\omega) = \int_0^t \gamma(s)\sigma^2(X_{s-}(\omega))\beta_s^2(\omega) ds + \int_0^t \int_{y\neq 0} \left(\sqrt{\bar{\pi}(s, X_{s-}(\omega), y)} - \sqrt{\pi(s, X_{s-}(\omega), y)}\right)^2 dy ds.$$

X_{s-} is a caglad process and hence is bounded for each ω on $[0,t]$ for every finite $t > 0$. The first integral in $H_t(\omega)$ is finite for every t, since $\gamma(s)$ is continuous and $\mu(x)$, $\bar{\mu}(x)$ are continuous. The second integral is also finite for every t due to our assumption in (ii). Hence $H_t(\omega) < \infty$ for every ω and t and H does not jump to infinity as defined in [13], III.5.8. This fact shows that $\bar{P}_{s,x}(H_t < \infty) = 1$ for all $t \geq s$, and it also shows that Hypothesis III.5.29 of [13] holds since $\sigma_{JS} = \infty$.

Lastly, by Theorem 3.1, $\bar{P}_{s,x}$ is the unique solution to the martingale problem for X associated with $(\bar{B}, \bar{C}, \bar{\nu})$. Hence, by Theorem III.2.40 of [13], local uniqueness also holds (see Definition III.2.35 in [13] for the definition of local uniqueness). Therefore, all the conditions stated in Theorem III.5.34 of [13] are satisfied, which, in turn, proves the claim. □

Remark 3.1. *If $\int_{(0,\infty)} \nu(s, d\tau) < \infty$ and $\int_{(0,\infty)} \bar{\nu}(s, d\tau) < \infty$ are bounded on $[0, t]$ for every finite $t > 0$, then condition (ii) in Theorem 3.2 is satisfied. To see this, note that*

$$
\begin{aligned}
&\int_{y\neq 0} \left(\sqrt{\bar{\pi}(s,x,y)} - \sqrt{\pi(s,x,y)}\right)^2 dy \\
&\leq \int_{y\neq 0} \bar{\pi}(s,x,y)dy + \int_{y\neq 0} \pi(s,x,y)dy \\
&= \int_{y\neq 0} \int_{(0,\infty)} \bar{p}(\tau,x,y)\bar{\nu}(s,d\tau)dy + \int_{y\neq 0} \int_{(0,\infty)} p(\tau,x,y)\nu(s,d\tau)dy \\
&= \int_{(0,\infty)} \int_{y\neq 0} \bar{p}(\tau,x,y)dy\bar{\nu}(s,d\tau) + \int_{(0,\infty)} \int_{y\neq 0} p(\tau,x,y)dy\nu(s,d\tau) \\
&\leq \int_{(0,\infty)} \bar{\nu}(s,d\tau) + \int_{(0,\infty)} \nu(s,d\tau).
\end{aligned}
$$

Remark 3.2. *When D is a Brownian motion with drift and T is a Lévy subordinator, X is a Lévy process. In this case, $\gamma(s)$, $\sigma(x)$ and the jump density $\pi(s, x, y)$ do not depend on s and x. Now suppose under both probability measures, X is a Lévy subordinate Brownian motion. Then condition (i) and (ii) in Theorem 3.2 coincide with the conditions in Theorem 33.1, [26] for general Lévy processes, which are also necessary.*

3.3 Verification of the Hellinger Condition

We next discuss how to verify the Hellinger condition in Theorem 3.2. We already know that it is satisfied under the condition in Remark 3.1. Below we consider the more difficult situation, where the condition in Remark 3.1 is violated. In many popular financial models, the subordinator has infinite activity.

Our next proposition shows that the Hellinger condition only depends on the behavior of

$\bar{\pi}(s,x,y)$ and $\pi(s,x,y)$ as y goes to 0.

Proposition 3.1. *For any $\delta > 0$,*

$$\int_{|y|>\delta} \left(\sqrt{\bar{\pi}(s,x,y)} - \sqrt{\pi(s,x,y)}\right)^2 dy \text{ is bounded on } [0,t] \times K$$

for any finite $t > 0$ and any compact set K of I.

Proof of Proposition 3.1. We have

$$\begin{aligned}
&\int_{|y|>\delta} \left(\sqrt{\bar{\pi}(s,x,y)} - \sqrt{\pi(s,x,y)}\right)^2 dy \\
&\leq \int_{|y|>\delta} \bar{\pi}(s,x,y)dy + \int_{|y|>\delta} \pi(s,x,y)dy \\
&\leq \int_{(0,\infty)} \int_{|y|>\delta} \bar{p}(\tau,x,y)dy\bar{\nu}(s,d\tau) + \int_{(0,\infty)} \int_{|y|>\delta} p(\tau,x,y)dy\nu(s,d\tau) \\
&\leq C_1 \int_{(0,\infty)} (\tau \wedge 1)\bar{\nu}(s,d\tau) + C_2 \int_{(0,\infty)} (\tau \wedge 1)\nu(s,d\tau),
\end{aligned}$$

for x on compacts, where C_1 and C_2 are positive constants that do not depend on s and x. The last inequality follows from Eq.(4.2) in [16]. Due to Assumption 2.2, the last line is bounded for s on $[0,t]$. □

The following proposition gives the implication of the Hellinger condition on the ratio of the jump density when y approaches 0.

Proposition 3.2. *Suppose for all $s \geq 0$, $x \in I$,*

$$\int_{(0,\infty)} \nu(s,d\tau) = \infty, \tag{3.7}$$

$$\lim_{y\to 0} \bar{\pi}(s,x,y)/\pi(s,x,y) \text{ exists.}$$

Then for any $\delta \in (0,\infty)$,

$$\int_{0<|y|<\delta} \pi(s,x,y)dy = \infty, \tag{3.8}$$

and condition (ii) in Theorem 3.2 implies that for all s and x,

$$\lim_{y\to 0} \bar{\pi}(s,x,y)/\pi(s,x,y) = 1.$$

Proof of Proposition 3.2. From Remark 2.2, (3.7) implies that

$$\int_{y\neq 0} \pi(s,x,y)dy = \infty.$$

Since $\Pi(s,x,dy)$ is a Lévy-type measure (see (2.6)), the above equation implies (3.8).

Now suppose the limit is not equal to 1. Then there is some $\delta_0 > 0$, such that for some $\epsilon > 0$,

$$\int_{0<|y|<\delta_0} \left(\sqrt{\bar{\pi}(s,x,y)} - \sqrt{\pi(s,x,y)}\right)^2 dy = \int_{0<|y|<\delta_0} \left(\sqrt{\frac{\bar{\pi}(s,x,y)}{\pi(s,x,y)}} - 1\right)^2 \pi(s,x,y)dy$$
$$\geq \epsilon \int_{0<|y|<\delta_0} \pi(s,x,y)dy = \infty.$$

Hence the limit must be 1 for all s and x. □

The asymptotic behavior of $\pi(s,x,y)$ as y goes to 0 is hard to obtain in general, as the diffusion transition probability density $p(\tau,x,y)$ is generally unknown and even it is known, it may not be easy to directly calculate the asymptotics for $\pi(s,x,y)$ when $y \to 0$. Our key idea is to look for another explicit positive function $q(\tau,x,y)$ that is close enough to $p(\tau,x,y)$ in small time, such that the asymptotics for $\int_{(0,\infty)} q(\tau,x,y)\nu(s,d\tau)$ is easy to calculate. We make the following assumption on $q(\tau,x,y)$.

Assumption 3.1. *Suppose there exist positive functions $q(\tau,x,y), \bar{q}(\tau,x,y)$ defined on $[0,\infty)\times I\times I$ that are jointly continuous in τ,x,y. Further suppose that there is some $\Delta \in (0,\infty)$. For any compact $\mathcal{K} \subseteq I\times I$, there are positive constants C and $\bar{C}$ such that for $(x,y)\in\mathcal{K}$, $0<\tau<\Delta$,*

$$|p(\tau,x,y) - q(\tau,x,y)| \leq q(\tau,x,y)C\tau, \tag{3.9}$$

$$|\bar{p}(\tau,x,y) - \bar{q}(\tau,x,y)| \leq \bar{q}(\tau,x,y)\bar{C}\tau, \tag{3.10}$$

We also assume that for any compact $K \subseteq I$, there is some $\delta > 0$, such that $\int_{|y|<\delta} q(\tau,x,x+y)dy$,

$\int_{|y|<\delta} \bar{q}(\tau, x, x+y)dy$ *are bounded for* $\tau \in (0, \Delta)$ *and* $x \in K$.

The next theorem shows that to verify the Hellinger condition in Theorem 3.2, it is sufficient to verify a Hellinger-type condition for two measures defined by q and $\bar{q}$.

Theorem 3.3. *Under Assumption 3.1, for any finite* $t > 0$ *and any compact set* K *of* I*, if*

$$\int_{0<|y|<\delta} \left| \int_{(0,\Delta)} q(\tau, x, x+y)\nu(s, d\tau) - \int_{(0,\Delta)} \bar{q}(\tau, x, x+y)\bar{\nu}(s, d\tau) \right| dy \text{ is bounded on } [0, t] \times K,$$

where δ *is the one chosen in Assumption 3.1, then condition (ii) in Theorem 3.2 is satisfied.*

Proof of Theorem 3.3. By Proposition 3.1, to show condition (ii) in Theorem 3.2 holds, we only need to show for any finite $t > 0$ and any compact set K of I, there is some $\delta \in (0, \infty)$ such that

$$\int_{0<|y|<\delta} \left(\sqrt{\bar{\pi}(s, x, y)} - \sqrt{\pi(s, x, y)} \right)^2 dy \text{ is bounded on } [0, t] \times K.$$

We use the δ in Assumption 3.1. Notice that

$$\begin{aligned}
&\int_{0<|y|<\delta} \left(\sqrt{\bar{\pi}(s, x, y)} - \sqrt{\pi(s, x, y)} \right)^2 dy \leq \int_{0<|y|<\delta} |\bar{\pi}(s, x, y) - \pi(s, x, y)|\, dy \\
&\leq \int_{0<|y|<\delta} \left| \int_{(0,\Delta)} p(\tau, x, x+y)\nu(s, d\tau) - \int_{(0,\Delta)} \bar{p}(\tau, x, x+y)\bar{\nu}(s, d\tau) \right| dy \\
&+ \int_{0<|y|<\delta} \left| \int_{[\Delta,\infty)} p(\tau, x, x+y)\nu(s, d\tau) - \int_{[\Delta,\infty)} \bar{p}(\tau, x, x+y)\bar{\nu}(s, d\tau) \right| dy \\
&\leq \int_{0<|y|<\delta} \left| \int_{(0,\Delta)} q(\tau, x, x+y)\nu(s, d\tau) - \int_{(0,\Delta)} \bar{q}(\tau, x, x+y)\bar{\nu}(s, d\tau) \right| dy \\
&+ \int_{0<|y|<\delta} \int_{(0,\Delta)} q(\tau, x, x+y)C\tau\nu(s, d\tau)dy \\
&+ \int_{0<|y|<\delta} \int_{(0,\Delta)} \bar{q}(\tau, x, x+y)\bar{C}\tau\bar{\nu}(s, d\tau)dy \\
&+ \int_{0<|y|<\delta} \int_{[\Delta,\infty)} p(\tau, x, x+y)\nu(s, d\tau)dy + \int_{0<|y|<\delta} \int_{[\Delta,\infty)} \bar{p}(\tau, x, x+y)\bar{\nu}(s, d\tau)dy.
\end{aligned}$$

The last inequality is obtained using (3.9) and (3.10). The first term is bounded on $[0, t] \times K$ by our assumption. Next we analyze the second term. Due to Assumption 3.1, $\int_{0<|y|<\delta} q(\tau, x, x+y)dy$

is bounded for $x \in K$ and $\tau \in (0, \Delta)$. Then

$$\begin{aligned}
&\int_{0<|y|<\delta} \int_{(0,\Delta)} q(\tau, x, x+y) C\tau\nu(s, d\tau) dy \\
&= C \int_{(0,\Delta)} \int_{0<|y|<\delta} q(\tau, x, x+y) dy \tau\nu(s, d\tau) \le C' \int_{(0,\Delta)} \tau\nu(s, d\tau)
\end{aligned}$$

for some positive constant C'. Since $\nu(s, d\tau)$ satisfies Assumption 2.2, the second term is bounded on $[0, t] \times K$. Similarly, one can show the remaining terms are bounded on $[0, t] \times K$ (for the fourth and fifth term, use Eq.(4.2) in [16]). Together we have shown that the Hellinger condition holds. □

We make some further assumptions on q and discuss the limit of the ratio of the measures defined by p and q when y tends to 0.

Assumption 3.2. *Suppose Assumption 3.1 is in force, and further assume that for all $x, y \in I$,*

$$\lim_{\tau \to \infty} q(\tau, x, y) = s(y), \ \lim_{\tau \to \infty} \bar{q}(\tau, x, y) = \bar{s}(y),$$

where $s(y), \bar{s}(y)$ are finite and continuous on I.

Theorem 3.4. *Under Assumption 3.2, suppose for all $s \ge 0$, (3.7) holds. Then for any $s \ge 0$, $x \in I$ and any $\delta > 0$, we have*

$$\lim_{y \to 0} \int_{(0,\infty)} p(\tau, x, x+y) \nu(s, d\tau) = \infty, \tag{3.11}$$

$$\lim_{y \to 0} \frac{\int_{(0,\delta)} p(\tau, x, x+y) \nu(s, d\tau)}{\int_{(0,\infty)} p(\tau, x, x+y) \nu(s, d\tau)} = 1, \ \lim_{y \to 0} \frac{\int_{(0,\delta)} q(\tau, x, x+y) \nu(s, d\tau)}{\int_{(0,\infty)} q(\tau, x, x+y) \nu(s, d\tau)} = 1, \tag{3.12}$$

$$\lim_{y \to 0} \frac{\int_{(0,\delta)} p(\tau, x, x+y) \nu(s, d\tau)}{\int_{(0,\delta)} q(\tau, x, x+y) \nu(s, d\tau)} = 1, \ \lim_{y \to 0} \frac{\int_{(0,\infty)} p(\tau, x, x+y) \nu(s, d\tau)}{\int_{(0,\infty)} q(\tau, x, x+y) \nu(s, d\tau)} = 1. \tag{3.13}$$

Proof of Theorem 3.4. We first prove that (3.7) implies (3.11). Proposition 3.2 shows that for any $\delta > 0$, (3.8) holds. If the limit of $\int_{(0,\infty)} p(\tau, x, x+y) \nu(s, d\tau)$ as $y \to 0$ is finite, then clearly (3.8) is finite for some $\delta_0 > 0$, which is a contradiction.

Next we show that

$$\lim_{y\to 0}\frac{\int_{(0,\delta)} p(\tau,x,x+y)\nu(s,d\tau)}{\int_{(0,\infty)} p(\tau,x,x+y)\nu(s,d\tau)}=1$$

for any $\delta>0$. Notice that, for the class of diffusions we consider, it is shown in [25] that

$$\lim_{\tau\to\infty} p(\tau,x,y)=\frac{m(y)}{m(I)}<\infty,$$

where $m(y)$ is the diffusion speed density given by

$$m(y)=\frac{2}{\sigma^2(y)}\exp\left(\int^y \frac{2\mu(z)}{\sigma^2(z)}dz\right),$$

and $m(I)=\int_I m(y)dy$. Clearly $m(y)/m(I)$ is continuous on I. This, together with the continuity of $p(\tau,x,x+y)$ in (τ,y), implies that $p(\tau,x,x+y)$ is bounded for $\tau>\delta$ and $y\in[0,\xi]$ for some $\xi>0$. We also have $\int_{[\delta,\infty)}\nu(s,d\tau)<\infty$. Hence, applying the Dominated Convergence Theorem,

$$\lim_{y\to 0}\int_{[\delta,\infty)} p(\tau,x,x+y)\nu(s,d\tau)=\int_{[\delta,\infty)}\lim_{y\to 0} p(\tau,x,x+y)\nu(s,d\tau)=\int_{[\delta,\infty)} p(\tau,x,x)\nu(s,d\tau),$$

which is finite. (3.11) implies

$$\lim_{y\to 0}\frac{\int_{[\delta,\infty)} p(\tau,x,x+y)\nu(s,d\tau)}{\int_{(0,\infty)} p(\tau,x,x+y)\nu(s,d\tau)}=0,$$

and consequently the first equality in (3.12) holds.

Fix an interval $[-\xi,\xi]$ for y. Assumption 3.1 implies that

$$\lim_{\tau\to 0}\frac{p(\tau,x,x+y)}{q(\tau,x,x+y)}=1$$

uniformly for y on $[-\xi,\xi]$. Hence, for any $\epsilon>0$, there exists some $\delta>0$, such that for any $y\in[-\xi,\xi]$,

$$1-\epsilon<\frac{p(\tau,x,x+y)}{q(\tau,x,x+y)}<1+\epsilon \quad \text{if } \tau<\delta.$$

Moving $q(\tau,x,x+y)$ to the LHS and RHS in the above equation and integrating w.r.t. $\nu(s,d\tau)$ for

τ on $(0, \delta)$ gives us

$$1 - \epsilon < \frac{\int_{(0,\delta)} p(\tau, x, x + y)\nu(s, d\tau)}{\int_{(0,\delta)} q(\tau, x, x + y)\nu(s, d\tau)} < 1 + \epsilon$$

for any $y \in [-\xi, \xi]$. Now letting $y \to 0$, we have

$$1 - \epsilon \leq \lim_{y \to 0} \frac{\int_{(0,\delta)} p(\tau, x, x + y)\nu(s, d\tau)}{\int_{(0,\delta)} q(\tau, x, x + y)\nu(s, d\tau)} \leq 1 + \epsilon.$$

Since ϵ is arbitrary, we obtain the first equality in (3.13). This equality further implies that $\lim_{y\to 0} \int_{(0,\delta)} q(\tau, x, x + y)\nu(s, d\tau) = \infty$ due to (3.11). We further have $\lim_{y\to 0} \int_{(0,\infty)} p(\tau, x, x + y)\nu(s, d\tau) = \infty$ as the integral on $(0, \infty)$ is greater than the integral on $(0, \delta)$.

Under Assumption 3.2, applying the argument that was used for proving the first equality in (3.12) shows the second one in (3.12).

The second equality in (3.13) can be obtained from the equalities in (3.12) and the first equality in (3.13). □

Combining Proposition 3.2 and Theorem 3.4, we obtain the following.

Corollary 3.1. *Suppose Assumption 3.2 holds. Further assume that for all $s \geq 0$ and $x \in I$,*

$$\int_{(0,\infty)} \bar{\nu}(s, d\tau) = \infty, \quad \int_{(0,\infty)} \nu(s, d\tau) = \infty,$$
$$\lim_{y\to 0} \int_{(0,\infty)} \bar{q}(\tau, x, x + y)\bar{\nu}(s, d\tau) \Big/ \int_{(0,\infty)} q(\tau, x, x + y)\nu(s, d\tau) \text{ exists (it could be infinite).}$$

Then condition (ii) in Theorem 3.2 implies that for all s and x, and any $0 < \delta \leq \infty$,

$$\lim_{y\to 0} \int_{(0,\delta)} \bar{q}(\tau, x, x + y)\bar{\nu}(s, d\tau) \Big/ \int_{(0,\delta)} q(\tau, x, x + y)\nu(s, d\tau) = 1.$$

The remaining question is how to find $q(\tau, x, y)$. The problem can be solved for a large class of diffusions that are commonly used in financial modelling. This class satisfies Assumption 2.1 and some additional regularity conditions imposed in [1, 2]. For simplicity, we do not restate those conditions here but refer readers to the Appendix of [1] for details. Popular diffusions in finance that fulfil these conditions include, but are not limited to, Brownian motion and many mean-reverting diffusions, such as the Ornstein-Uhlenbeck process, the CIR process and the 3/2 process.

For this class of diffusions, [1, 2] derives a small time expansion for its transition density $p(\tau, x, y)$, i.e., there exists $\Delta > 0$ such that for $0 < \tau \leq \Delta$,

$$p(\tau, x, y) = p_0(\tau, x, y) \left(1 + \sum_{n=1}^{\infty} d_n(x, y) \frac{\tau^n}{n!}\right),$$

where $d_n(x, y)$ are the expansion coefficients which are infinitely continuously differentiable in x and y, and $p_0(\tau, x, y)$ is given as follows: Define

$$\gamma(x) = \int_{x_0}^{x} \frac{1}{\sigma(u)} du \quad \text{for some } x_0 \in (l, r), \tag{3.14}$$

which is an increasing function. Let

$$\tilde{\mu}(u) = \frac{\mu(\gamma^{-1}(u))}{\sigma(\gamma^{-1}(u))} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(\gamma^{-1}(u)) \quad \text{for } u \in (\gamma(l), \gamma(r)). \tag{3.15}$$

Then

$$p_0(\tau, x, y) = \frac{\tau^{-\frac{1}{2}} \phi\left(\frac{\gamma(y)-\gamma(x)}{\sqrt{\tau}}\right) \exp\left(\int_{\gamma(x)}^{\gamma(y)} \tilde{\mu}(u) du\right)}{\sigma(y)}, \tag{3.16}$$

where $\phi(\cdot)$ is the standard Normal density function. We now show that $p_0(\tau, x, y)$ satisfies Assumptions 3.1 and 3.2, and hence, we can set $q(\tau, x, y) = p_0(\tau, x, y)$.

Proposition 3.3. *The function $q(\tau, x, y) = p_0(\tau, x, y)$ defined by (3.16) satisfies the Assumptions 3.1 and 3.2.*

Proof of Proposition 3.3. Note that $\sum_{n=1}^{\infty} d_n(x, y) \frac{\tau^n}{n!}$ is continuous in (τ, x, y) and

$$\lim_{\tau \to 0} \frac{1}{\tau} \sum_{n=1}^{\infty} d_n(x, y) \frac{\tau^n}{n!} = d_1(x, y),$$

where $d_1(x, y)$ is continuous. Therefore $(p(\tau, x, y) - q(\tau, x, y))/(\tau q(\tau, x, y))$ is bounded for $\tau \in (0, \Delta)$ and (x, y) on compacts. So (3.9) is valid.

For a compact set $K \subseteq I$, choose a $\delta \in (0, \infty)$ that is small enough such that $\mathcal{Y}_\delta := \{y : |y - x| < \delta, x \in K\}$ is contained in a compact subset of I. Now consider $\int_{|y-x|<\delta} q(\tau, x, y) dy$. We want to show it is bounded for $\tau \in (0, \Delta)$ and $x \in K$. Note that $\exp\left(\int_{\gamma(x)}^{\gamma(y)} \tilde{\mu}(u) du\right)$ is bounded for $x \in K$

and $y \in \mathcal{Y}_\delta$. Hence for $\tau \in (0, \Delta)$ and $x \in K$,

$$\begin{aligned}\int_{|y-x|<\delta} q(\tau, x, y) dy &\le C \int_{|y-x|<\delta} \frac{1}{\sigma(y)\sqrt{\tau}} \phi\left(\frac{\gamma(y)-\gamma(x)}{\sqrt{\tau}}\right) dy \\ &= C \int_{(\gamma(x-\delta), \gamma(x+\delta))} \frac{1}{\sqrt{\tau}} \phi\left(\frac{u-\gamma(x)}{\sqrt{\tau}}\right) du \quad (u = \gamma(y)) \\ &\le \sqrt{2\pi} C\end{aligned}$$

for some positive constant C. So Assumption 3.1 is verified.

Finally, since $\lim_{\tau \to \infty} q(\tau, x, y) = 0$, Assumption 3.2 is satisfied. □

For a large family of Lévy measures, the asymptotic behavior of $\int_{(0,\delta)} q(\tau, x, x+y)\nu(s, d\tau)$ as $y \to 0$ can be obtained explicitly, and conditions in Theorem 3.2 reduce to explicit restrictions on the model parameters. An example is given in the next section.

4 An Example

An additive subordinate CIR (ASubCIR) process is obtained by applying additive subordination to the CIR diffusion. It is used for modelling commodity prices in [16], which shows such process is able to perform well empirically and also provides analytical tractability for derivatives pricing. [16] only considers modelling using the ASubCIR process under the risk-neutral measure. Below we wish to obtain explicit conditions under which the ASubCIR process remains to be an ASubCIR process under locally equivalent measure changes. Using such measure transformation, the model remains to be tractable under the physical measure, which makes joint estimation using time series data of the underlying asset and derivative prices much easier.

In the CIR diffusion, $l = 0$ and $r = \infty$,

$$\mu(x) = \kappa(\theta - x), \ \sigma(x) = \sigma\sqrt{x},$$

where $\kappa, \theta, \sigma > 0$. We impose the Feller condition $2\kappa\theta \ge \sigma^2$ so that the CIR diffusion cannot hit 0. Hence $I = (0, \infty)$.

We consider the regularized Sato-type tempered stable subordinators developed in [16], which is used for calibration there. This is a parsimonious family of subordinators with self-similarity satisfying Assumption 2.1. In this family,

$$\gamma(s) = \gamma\rho(s+t_0)^{\rho-1},\ \nu(s,d\tau) = C\rho(s+t_0)^{\rho\alpha-1}\tau^{-\alpha-1}e^{-\eta\tau(s+t_0)^{-\rho}}(\alpha+\eta\tau(s+t_0)^{-\rho})d\tau,$$

where $\gamma \geq 0$, $0 < \alpha < 1$, $\eta > 0$, $\rho > 0$, $t_0 \geq 0$ if $\rho \geq 1$ and $t_0 > 0$ if $0 < \rho < 1$.

For the CIR process, using (3.14), (3.15) and (3.16), $q(\tau, x, y)$ is given by

$$q(\tau,x,y) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}y^{-\frac{1}{2}}\left(\frac{y}{x}\right)^{-1/4+\kappa\theta/\sigma^2}\exp\left(-\frac{2(\sqrt{y}-\sqrt{x})^2}{\sigma^2\tau}-\frac{\kappa(y-x)}{\sigma^2}\right).$$

Our next proposition obtains explicit constraints on the parameters that are equivalent to the conditions in Theorem 3.2. We add bar on the alphabets to denote model parameters under the new probability measure $\bar{P}_{s,x}$, and the Feller condition $2\bar{\kappa}\bar{\theta} \geq \bar{\sigma}^2$ is imposed. The results show that κ, θ, η and C can be freely changed in the admissible parameter space while the other parameters obey the restrictions in (4.1).

Proposition 4.1. *Condition (i) and (ii) in Theorem 3.2 are equivalent to the following:*

$$\rho = \bar{\rho},\ \alpha = \bar{\alpha},\ t_0 = \bar{t}_0,\ \gamma\sigma^2 = \bar{\gamma}\bar{\sigma}^2,\ C\sigma^{2\alpha} = \bar{C}\bar{\sigma}^{2\alpha}. \tag{4.1}$$

Under these conditions, for any $s \geq 0$, $x \in I$, $\bar{P}_{s,x}|_{\mathcal{F}_{s,t}} \sim P_{s,x}|_{\mathcal{F}_{s,t}}$ for every $t \geq s$.

Proof of Proposition 4.1. Condition (i) in Theorem 3.2 requires that for almost all $s \geq 0$ and $x > 0$

$$\gamma\rho(s+t_0)^{\rho-1}\sigma^2 x = \bar{\gamma}\bar{\rho}(s+\bar{t}_0)^{\bar{\rho}-1}\bar{\sigma}^2 x \tag{4.2}$$

Continuity of both sides in s and x implies this is true for all $s \geq 0$ and $x > 0$. Obviously (4.2) cannot hold if only one of γ and $\bar{\gamma}$ is zero. Now suppose $\gamma \neq 0$ and $\bar{\gamma} \neq 0$. Then re-arranging terms, (4.2) becomes

$$\frac{\gamma\rho\sigma^2}{\bar{\gamma}\bar{\rho}\bar{\sigma}^2} = \frac{(s+\bar{t}_0)^{\bar{\rho}-1}}{(s+t_0)^{\rho-1}},$$

for all $s \geq 0$. Since the LHS is a constant and the RHS depends on s, we must have $\rho = \bar{\rho}$ and $t_0 = \bar{t}_0$. Hence we also have $\gamma\sigma^2 = \bar{\gamma}\bar{\sigma}^2$. When $\gamma = \bar{\gamma} = 0$, (4.2) is satisfied automatically and no conditions need to be imposed on the other parameters.

We next consider condition (ii) in Theorem 3.2. We can write

$$\nu(s, d\tau) = \nu_1(s, d\tau) + \nu_2(s, d\tau),$$

where

$$\nu_1(s, \tau) = C\rho\alpha(s + t_0)^{\rho\alpha - 1}\tau^{-\alpha-1}e^{-\eta\tau(s+t_0)^{-\rho}},$$

$$\nu_2(s, \tau) = C\rho\eta(s + t_0)^{\rho\alpha-\rho-1}\tau^{-\alpha}e^{-\eta\tau(s+t_0)^{-\rho}}.$$

We first obtain the asymptotic behavior of $\int_{(0,\delta)} q(\tau, x, y)\nu_1(s, d\tau)$. Define

$$g(x, y) = y^{-\frac{1}{2}}\left(\frac{y}{x}\right)^{-1/4+\kappa\theta/\sigma^2}\exp\left(-\frac{\kappa(y - x)}{\sigma^2}\right), \ h(s) = C\rho\alpha(s + t_0)^{\rho\alpha-1}.$$

Then let $u = 2(\sqrt{y} - \sqrt{x})^2/(\sigma^2\tau)$, we have

$$\begin{aligned}
&\int_{(0,\delta)} q(\tau, x, y)\nu_1(s, d\tau) = g(x, y)\frac{|\sqrt{y} - \sqrt{x}|}{\sigma^2\sqrt{\pi}}\int_{\frac{2(\sqrt{y}-\sqrt{x})^2}{\sigma^2\delta}}^{\infty} u^{-\frac{3}{2}}e^{-u}\nu_1\left(s, \frac{2(\sqrt{y} - \sqrt{x})^2}{\sigma^2 u}\right)du\\
&= g(x, y)h(s)\frac{\sigma^{2\alpha}}{2^{\alpha+1}\sqrt{\pi}}\left|\sqrt{y} - \sqrt{x}\right|^{-2\alpha-1}\int_{\frac{2(\sqrt{y}-\sqrt{x})^2}{\sigma^2\delta}}^{\infty} u^{\alpha-\frac{1}{2}}e^{-u}\exp\left(-\eta\frac{2(\sqrt{y} - \sqrt{x})^2}{\sigma^2 u}(s + t_0)^{-\rho}\right)du\\
&\sim h(s)\frac{\sigma^{2\alpha}\Gamma(\alpha + \frac{1}{2})}{2^{\alpha+1}\sqrt{\pi x}}\left|\sqrt{y} - \sqrt{x}\right|^{-2\alpha-1} \quad \text{as } y \to x,
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function. Similar calculations show that as $y \to 0$, $\int_{(0,\delta)} q(\tau, x, y)\nu_2(s, d\tau)$ either converges to a finite value or to ∞ more slowly than $\int_{(0,\delta)} q(\tau, x, y)\nu_1(s, d\tau)$. Therefore to analyze the limit of $\int_{(0,\delta)} \bar{q}(\tau, x, x + y)\bar{\nu}(s, d\tau)\Big/\int_{(0,\delta)} q(\tau, x, x + y)\nu(s, d\tau)$ as $y \to 0$, we only need to consider the integral with $\bar{\nu}_1$ and ν_1 as the measure. Corollary 3.1 implies that

$$\alpha = \bar{\alpha}, \ C\rho\alpha(s + t_0)^{\rho\alpha-1}\frac{\sigma^{2\alpha}\Gamma(\alpha + \frac{1}{2})}{2^{\alpha+1}\sqrt{\pi}} = \bar{C}\bar{\rho}\bar{\alpha}(s + \bar{t}_0)^{\bar{\rho}\bar{\alpha}-1}\frac{\bar{\sigma}^{2\bar{\alpha}}\Gamma(\bar{\alpha} + \frac{1}{2})}{2^{\bar{\alpha}+1}\sqrt{\pi}}, \tag{4.3}$$

It is not difficult to see that (4.3) also implies that for any finite $t > 0$ and compact subset K of I.

$$\int_{0<|y|<\delta}\left|\int_{(0,\Delta)} q(\tau, x, x + y)\nu_1(s, d\tau) - \int_{(0,\Delta)} \bar{q}(\tau, x, x + y)\bar{\nu}_1(s, d\tau)\right| \text{ is bounded on } [0, t] \times K,$$

where δ is the one chosen in Assumption 3.1. Since $0 < \alpha < 1$, $\int_{(0,\infty)} \nu_2(s, d\tau) < \infty$ is bounded on $[0, t]$ for every finite $t > 0$. Therefore using the arguments in Remark 3.1 shows

$$\int_{0<|y|<\delta} \left| \int_{(0,\Delta)} q(\tau, x, x+y)\nu_2(s, d\tau) - \int_{(0,\Delta)} \bar{q}(\tau, x, x+y)\bar{\nu}_2(s, d\tau) \right| \text{ is bounded on } [0, t] \times K,$$

for any finite $t > 0$ and compact subset K of I. Together Theorem 3.3 applies and condition (ii) in Theorem 3.2 is equivalent to (4.3).

Now using similar arguments as before, (4.3) is equivalent to $\alpha = \bar{\alpha},\ \rho = \bar{\rho},\ t_0 = \bar{t}_0,\ C\sigma^{2\alpha} = \bar{C}\bar{\sigma}^{2\alpha}$. This together with the conditions derived from (4.2) gives us (4.1). □

When the additive subordinator is in the family of regularized Sato-type tempered stable subordinators, and when the background diffusion is e.g., Brownian motion, Ornstein-Uhlenbeck, CEV-type diffusion with linear or nonlinear mean-reverting drift, one can perform similar calculations and reduce conditions in Theorem 3.2 to explicit restrictions on the model parameters.

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