On-Line Companion to “Time Changed Markov Processes in Unified Credit-Equity Modeling”

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A Proofs

A.1 Supplement to the Proof of Theorem 4.3

For $f \in C_c^2((0,\infty))$ we have for the integral term in (A.3):

$$\int_{(0,\infty)} (\mathcal{P}_s f(x) - f(x))\nu(ds) = \int_{(0,\infty)} \left(\int_{(0,\infty)} p(s;x,y) f(y) dy - f(x) \right) \nu(ds)$$

$$= \int_{(0,\infty)} \left\{ \int_{(0,\infty)} p(s;x,y) \left[\left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{df}{dx}(x) \right) \right.\right.$$

$$\left.\left. + f(x) + \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{df}{dx}(x) \right] dy - f(x) \right\} \nu(ds)$$

$$= \int_{(0,\infty)} \left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{df}{dx}(x) \right) \left(\int_{(0,\infty)} p(s;x,y)\nu(ds) \right) dy$$

$$- \left(\int_{(0,\infty)} \left(1 - \int_{(0,\infty)} p(s;x,y)dy \right) \nu(ds) \right) f(x)$$

$$+ \left(\int_{(0,\infty)} \left(\int_{\{y>0:|y-x|\le 1\}} (y-x) p(s;x,y) dy \right) \nu(ds) \right) \frac{df}{dx}(x)$$

$$= \int_{(0,\infty)} \left(f(y) - f(x) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{df}{dx}(x) \right) \Pi(x,dy)$$

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$$- \left(\int_{(0,\infty)} P_s(x, \{\Delta\}) \nu(ds) \right) f(x)$$

$$+ \left(\int_{(0,\infty)} \left(\int_{\{y>0:|y-x|\le 1\}} (y-x) p(s;x,y) dy \right) \nu(ds) \right) \frac{df}{dx}(x).$$

Substituting this result in (A.3), we arrive at Eqs.(4.5)-(4.9).

A.2 Proof of Theorem 8.2

(i) We present the proof of the spectral expansion by directly inverting the Laplace transform for the transition density in section 6 by applying the Cauchy Residue Theorem. When $\mu + b > 0$, the Green's function $G_s(x,y)$ Eq.(6.3) is given by:

$$G_s(x,y) = \frac{\Gamma\left(\frac{1+\nu}{2} - \varkappa(s)\right)}{(\mu+b)\Gamma(1+\nu)} x^{\frac{1}{2}-c+\beta} y^{c-\frac{3}{2}-\beta} e^{-\frac{A}{2}\left(x^{-2\beta} - y^{-2\beta}\right)}$$

$$\times \begin{cases} M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right) W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & x \le y \\ W_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right) M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & y \le x \end{cases} .$$

The only singularities of the Green's function are simple poles of the Gamma function $\Gamma(\nu/2 + 1/2 - \varkappa(s))$ at $\nu/2 + 1/2 - \varkappa(s) = -n + 1$, $n = 1, 2, ...$, i.e., at $s = -\lambda_n$ with $\lambda_n = \omega n + \xi$ for $n = 1, 2, ...$. Applying the Cauchy Residue Theorem, in this case the Laplace inversion integral is equal to the sum of residues at the poles:

$$p(t;x,y) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st} G_s(x,y) ds = \sum_{n=1}^{\infty} \mathrm{Res}_{s=-\lambda_n}\left(e^{st} G_s(x,y)\right).$$

The residues are:

$$\mathrm{Res}_{s=-\lambda_n}\left(e^{st} G_s(x,y)\right) = \left. \frac{\omega(-1)^{n+1} e^{st} G_s(x,y)}{(n-1)!\Gamma\left(\frac{1+\nu}{2} - \varkappa(s)\right)} \right|_{s=-\omega n - \xi} .$$

Substituting this into the sum, we have:

$$p(t;x,y) = \sum_{n=1}^{\infty} e^{-(\omega n + \xi)t} \frac{\omega(-1)^{n-1}}{(n-1)!(\mu+b)\Gamma(1+\nu)} x^{\frac{1}{2}-c+\beta} y^{c-\frac{3}{2}-\beta} e^{-\frac{A}{2}\left(x^{-2\beta} - y^{-2\beta}\right)}$$

$$\times \begin{cases} M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ax^{-2\beta}\right) W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & x \le y \\ W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ax^{-2\beta}\right) M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}\left(Ay^{-2\beta}\right), & y \le x \end{cases} .$$

When $\varkappa = \frac{\nu}{2} + n - \frac{1}{2}$, $n = 1, 2, ...$, the Whittaker functions $M_{\varkappa,\frac{\nu}{2}}(x)$ and $W_{\varkappa,\frac{\nu}{2}}(x)$ become linearly dependent and reduce to the generalized Laguerre polynomials (see Buchholz (1969), p.214):

$$M_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}(x) = \frac{(n-1)!\Gamma(1+\nu)}{\Gamma(\nu+n)} e^{-x/2} x^{\frac{\nu+1}{2}} L_{n-1}^{(\nu)}(x),$$

$$W_{\frac{\nu}{2}+n-\frac{1}{2},\frac{\nu}{2}}(x) = (n-1)!(-1)^{n-1} e^{-x/2} x^{\frac{\nu+1}{2}} L_{n-1}^{(\nu)}(x).$$

Substituting this result in the sum we obtain the spectral representation of the transition probability density:

$$p(t;x,y)=\mathfrak{m}\left(y\right)\sum_{n=1}^{\infty}e^{-(\omega n+\xi)t}\frac{A^{\nu}(\mu+b)(n-1)!}{\Gamma(\nu+n)}xye^{-A\left(x^{-2\beta}+y^{-2\beta}\right)}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)L_{n-1}^{(\nu)}\left(Ax^{-2\beta}\right)$$

$$=\mathfrak{m}\left(y\right)\sum_{n=1}^{\infty}e^{-\lambda_n t}\varphi_n(x)\varphi_n(y)$$

with the eigenvalues and eigenfunctions (8.11).

(ii) When $\mu+b=0$, the Green's function $G_s(x,y)$ is given by:

$$G_s(x,y)=\frac{2}{a^2|\beta|}x^{\frac{1}{2}-c}y^{c-3/2-2\beta}\begin{cases} I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)K_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), & x\le y\\ K_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)I_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), & y\le x\end{cases}.$$

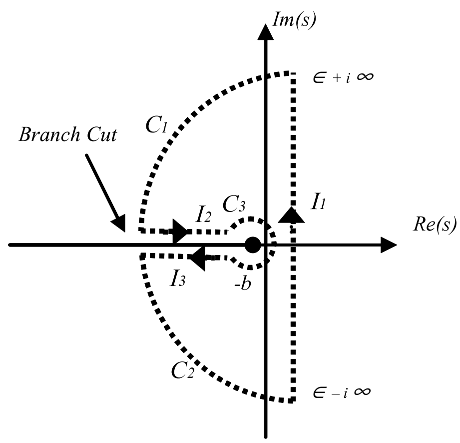

Figure 1: *The Bromwich Laplace inversion is over the vertical line I_1. C_1 and C_2 are the integration arcs at infinity, C_3 is the arc around the branching point. I_2 and I_3 are the lines of integration at each side of the branch cut.*

The only singularity of the Green's function is a branching point at $s=-b$. We place the branch cut along the negative real line from $-\infty$ to $-b$. We now use the Cauchy's Theorem to calculate the Laplace inversion integral. We consider a closed contour in Figure 1. Since the function is analytic inside the contour, the integral along the closed contour vanishes. On the other hand, the integral is equal to the sum of the integral along the line parallel to the imaginary axes in the Bromwich Laplace inversion integral, the integrals along the two arcs at infinity, the integrals along each side of the branch cut, and the integral along the arc around the branching point $s=-b$. We now show that the integrals along the arcs at infinity and along the arc around the branching point vanish. We do this by considering the asymptotics of the Green's function.

Let $s=\rho e^{i\theta}-b$ and $ds=i\rho e^{i\theta}d\theta$. The asymptotic value of the Bessel functions' products in the Green's function vanish when either $\rho\to 0$ or when $\rho\to\infty$ and $\theta\in(\pi/2,\pi)\cup(-\pi,-\pi/2)$:

$$e^{t\rho e^{i\theta}} I_\nu\left(\frac{x^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right) K_\nu\left(\frac{y^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right)\rho e^{i\theta}\longrightarrow 0 \text{ as } \rho\to 0.$$

To show this, note that asymptotically for $\rho\to 0$ we have:

$$I_\nu(a\sqrt{\rho})\approx\left(\frac{a}{2}\right)^\nu\frac{\rho^{\nu/2}}{\Gamma(\nu+1)}+\left(\frac{a}{2}\right)^{\nu+2}\frac{\rho^{\nu/2+1}}{\Gamma(\nu+2)}+\rho^{\nu/2}O\left(\rho^2\right) \text{ as } \rho\to 0,$$

$$K_\nu(b\sqrt{\rho})\approx\frac{1}{2}\left(\frac{b}{2}\right)^\nu\Gamma(-\nu)\rho^{\nu/2}+\frac{1}{2}\left(\frac{b}{2}\right)^{\nu+2}\frac{\Gamma(-\nu)\rho^{\nu/2+1}}{1+\nu}$$
$$+\frac{1}{2}\left(\frac{b}{2}\right)^{-\nu}\Gamma(\nu)\rho^{-\nu/2}+\frac{1}{2}\left(\frac{b}{2}\right)^{2-\nu}\frac{\Gamma(\nu)\rho^{1-\nu/2}}{1-\nu}+\left(\rho^{\nu/2}+\rho^{-\nu/2}\right)O\left(\rho^2\right) \text{ as } \rho\to 0,$$

and, hence,

$$\rho I_\nu(a\sqrt{\rho})K_\nu(b\sqrt{\rho})\approx\left(\frac{a}{b}\right)^\nu\frac{\rho}{2\nu}+\frac{1}{2}\left(\frac{ab}{4}\right)^\nu\frac{\Gamma(-\nu)}{\Gamma(1+\nu)}\rho^{\nu+1}+(1+\rho^\nu)O\left(\rho^2\right) \text{ as } \rho\to 0.$$

Likewise we have:

$$e^{t\rho e^{i\theta}} I_\nu\left(\frac{x^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right) K_\nu\left(\frac{y^{-\beta}\sqrt{2\rho}e^{i\theta/2}}{a|\beta|}\right)\rho e^{i\theta}\longrightarrow 0 \text{ as } \rho\to \infty.$$

To show this, note that asymptotically as $\rho\to\infty$ we have:

$$I_\nu\left(a\sqrt{\rho}e^{\frac{\theta}{2}i}\right)\approx\frac{1}{\sqrt{2\pi}}e^{a\sqrt{\rho}e^{\frac{\theta}{2}i}}a^{-\frac{1}{2}}\rho^{-\frac{1}{4}}e^{-\frac{\theta}{4}i} \text{ as } \rho\to\infty,$$

$$K_\nu\left(b\sqrt{\rho}e^{\frac{\theta}{2}i}\right)\approx\frac{1}{\sqrt{2}}\sqrt{\pi}e^{-b\sqrt{\rho}e^{\frac{\theta}{2}i}}b^{-\frac{1}{2}}\rho^{-\frac{1}{4}}e^{-\frac{\theta}{4}i} \text{ as } \rho\to\infty,$$

and, hence,

$$e^{t\rho e^{i\theta}}I_\nu\left(a\sqrt{\rho}e^{\frac{\theta}{2}i}\right)K_\nu\left(b\sqrt{\rho}e^{\frac{\theta}{2}i}\right)\approx\frac{1}{2}e^{\sqrt{\rho}e^{\frac{\theta}{2}i}(a-b)+\rho e^{i\theta}t}(ab)^{-\frac{1}{2}}\rho^{-\frac{1}{2}}e^{-\frac{\theta}{2}i} \text{ as } \rho\to\infty.$$

According to Eq.(6.3) for the Green's function, the argument of I_ν is at most as large as the argument of K_ν (i.e., $a\le b$ for both cases $x<y$ and $x>y$), and since for $\theta\in(\pi/2,\pi)\cup(-\pi,-\pi/2)$ we have that $\Re\left(e^{i\theta}\right)=\cos(\theta)<0$ and $\Re\left(e^{i\theta/2}(a-b)\right)=(a-b)\cos(\theta/2)\le 0$, the product vanishes as $\rho\to\infty$.

Thus, the integrals along the arcs at infinity and along the arc around the branching point vanish, and the Laplace inversion integral reduces to the integral of the jump across the branch cut (where we changed the integration variable according to $s=\left(\lambda e^{\pm\pi i}-b\right)$):

$$p(t;x,y)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}G_s(x,y)ds=\frac{1}{2\pi i}\int_0^\infty e^{-(\lambda+b)t}\left(G_{\lambda e^{-\pi i}-b}(x,y)-G_{\lambda e^{\pi i}-b}(x,y)\right)d\lambda.$$

Recall the following identities for Bessel functions:

$$I_\nu\left(e^{\pm\frac{\pi}{2}i}a\right)=e^{\pm\frac{\nu\pi}{2}}J_\nu\left(a\right),$$

$$K_\nu\left(e^{\pm\frac{\pi}{2}i}b\right)=\mp\frac{\pi i}{2}e^{\mp\frac{\nu\pi}{2}}\left(J_\nu\left(b\right)\mp iY_\nu\left(b\right)\right),$$

and

$$I_\nu\left(e^{-\frac{\pi}{2}i}a\right)K_\nu\left(e^{-\frac{\pi}{2}i}b\right)-I_\nu\left(e^{\frac{\pi}{2}i}a\right)K_\nu\left(e^{\frac{\pi}{2}i}b\right)$$

$$=\frac{\pi i}{2}J_\nu\left(a\right)\left(J_\nu\left(b\right)+iY_\nu\left(b\right)\right)+\frac{\pi i}{2}J_\nu\left(a\right)\left(J_\nu\left(b\right)-iY_\nu\left(b\right)\right)=\pi iJ_\nu\left(a\right)J_\nu\left(b\right).$$

Substituting the expression for the Green's function into the integral across the branch cut and using this identity, we arrive at the spectral expansion for the transition density:

$$p(t;x,y)=\frac{y^{2c-2-2\beta}}{a^2|\beta|}(yx)^{\frac{1}{2}-c}\int_0^\infty e^{-(\lambda+b)t}J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)d\lambda.$$

This completes the proof. □

A.3 Proof of Theorem 8.3

To compute the survival probability, we need to compute $\mathcal{P}_tf(x)$ for $f(x)=1$ (note that the semigroup is non-conservative due to killing (default), and so $(\mathcal{P}_t1)(x)\leq 1$). Since in this case the constants are not square-integrable with the speed density, we cannot apply the spectral theory. Instead, we first compute the resolvent $(\mathcal{R}_s1)(x)$ by integrating $f(x)=1$ with the Green's function, and then invert the Laplace transform, as outlined in section 6.

(i) For $\mu+b>0$ we have:

$$(\mathcal{R}_s1)(x)=\frac{\Gamma\left(\frac{1+\nu}{2}-\varkappa(s)\right)}{(\mu+b)\Gamma(1+\nu)}x^{\frac{1}{2}-c+\beta}e^{-\frac{A}{2}x^{-2\beta}}$$

$$\times\left\{M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)\int_x^\infty y^{c-\frac{3}{2}-\beta}e^{\frac{A}{2}y^{-2\beta}}W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right)dy\right.$$

$$\left.+W_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)\int_0^x y^{c-\frac{3}{2}-\beta}e^{\frac{A}{2}y^{-2\beta}}M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right)dy\right\}.$$

Using the integrals (B.11) and (B.12) for the Whittaker functions, we calculate the integrals in closed form:

$$\int_x^\infty y^{c-\frac{3}{2}-\beta}e^{\frac{A}{2}y^{-2\beta}}W_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right)dy=\frac{A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}}}{2|\beta|}\frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)}{\Gamma\left(\frac{s+\xi}{\omega}+1\right)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)}$$

$$-\frac{A^{\frac{1-\nu}{2}}x^{-2\beta-1}\Gamma\left(\nu\right)}{(2|\beta|-1)\Gamma\left(\frac{s+\xi}{\omega}+1\right)}\,{}_2F_2\left(\begin{matrix}1-\frac{1}{2|\beta|}, & 1+\frac{s+\xi}{\omega}-\nu\\ 2-\frac{1}{2|\beta|}, & 1-\nu\end{matrix};Ax^{-2\beta}\right)$$

$$-\frac{A^{\frac{1+\nu}{2}}x^{2c-2\beta}\Gamma\left(-\nu\right)}{(2|\beta|+2c)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)}\,{}_2F_2\left(\begin{matrix}1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega}\\ 2+\frac{c}{|\beta|}, & 1+\nu\end{matrix};Ax^{-2\beta}\right),$$

$$\int_0^x y^{c-\frac{3}{2}-\beta} e^{\frac{A}{2}y^{-2\beta}} M_{\varkappa(s),\frac{\nu}{2}}\left(Ay^{-2\beta}\right)dy = \frac{A^{\frac{1+\nu}{2}} x^{2c-2\beta}}{(2|\beta|+2c)}\, {}_2F_2\left(\begin{matrix} 1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega} \\ 2+\frac{c}{|\beta|}, & 1+\nu \end{matrix}; Ax^{-2\beta}\right).$$

Using the identity

$$\frac{\pi}{\sin(\pi\nu)} = -\Gamma(-\nu)\Gamma(1+\nu)$$

and Eq.(B.9), we obtain the resolvent $(\mathcal{R}_s 1)(x)$:

$$(\mathcal{R}_s 1)(x) = \frac{x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}}}{(\mu+b)}$$

$$\times \Bigg\{ M_{\varkappa(s),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)\Bigg[\frac{A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}}}{2|\beta|} \frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{s+\xi}{\omega}+1-\nu\right)}$$

$$-\frac{A^{\frac{1-\nu}{2}} x^{-2\beta-1}}{\nu\,(2|\beta|-1)}\, {}_2F_2\left(\begin{matrix} 1-\frac{1}{2|\beta|}, & 1+\frac{s+\xi}{\omega}-\nu \\ 2-\frac{1}{2|\beta|}, & 1-\nu \end{matrix}; Ax^{-2\beta}\right)\Bigg]$$

$$-M_{\varkappa(s),-\frac{\nu}{2}}\left(Ax^{-2\beta}\right) \frac{A^{\frac{1+\nu}{2}} x^{2c-2\beta}\Gamma(-\nu)}{\Gamma(1-\nu)(2|\beta|+2c)}\, {}_2F_2\left(\begin{matrix} 1+\frac{c}{|\beta|}, & 1+\frac{s+\xi}{\omega} \\ 2+\frac{c}{|\beta|}, & 1+\nu \end{matrix}; Ax^{-2\beta}\right)\Bigg\}$$

Now we can proceed by inverting the Laplace transform Eq.(6.8) by means of the Cauchy Residue Theorem similar to the proof of Theorem 8.2.

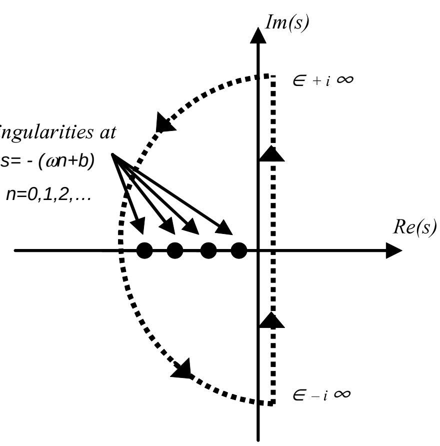

Figure 2: *Singularities enclosed by the integration contour are located at* $s = -(\omega n + b)$ *for* $n = 0, 1, 2...$

The only singularities of the resolvent $(\mathcal{R}_s 1)(x)$ are simple poles of the Gamma function $\Gamma\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right)$ in the first term at $\left(\frac{s+\xi}{\omega}-\frac{c}{|\beta|}\right) = -n$ for $n = 0, 1, 2, ...$, i.e., at $s = -(\omega n + b)$ for $n = 0, 1, 2, ...$, this is shown in Figure 2. The last two terms have no singularities and thus do not contribute to the Laplace inversion integral. By applying the Cauchy Residue Theorem, we reduce the Laplace transform inversion integral for the survival probability to the sum over the residues at the poles:

$$\mathbb{Q}(\zeta > t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{st}(\mathcal{R}_s 1)(x)ds = \frac{\omega A^{\frac{1-2c}{4|\beta|}-\frac{1}{2}} x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}}}{2|\beta|(\mu+b)} \frac{\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(1+\frac{c}{|\beta|}\right)}{\Gamma(\nu+1)}$$

$$\times \sum_{n=0}^{\infty} \frac{e^{-(b+\omega n)t}(-1)^{-n}}{\Gamma\left(1-\frac{1}{2|\beta|}-n\right)n!} M_{n+\left(\frac{1-2(c+|\beta|)}{4|\beta|}\right),\frac{\nu}{2}}\left(Ax^{-2\beta}\right).$$

Finally, using the identity

$$(a)_n = (-1)^n \Gamma(1-a)/\Gamma(1-a-n)$$

and writing the Whittaker function in terms of the confluent hypergeometric function ${}_1F_1$ we obtain the explicit result (8.13) for the survival probability.

(ii) For $\mu + b = 0$ the resolvent is:

$$(\mathcal{R}_s 1)(x) = \frac{2}{a^2|\beta|} x^{\frac{1}{2}-c} \left\{ K_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) \int_0^x y^{c-3/2-2\beta} I_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) dy \right.$$

$$\left. + I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) \int_x^\infty y^{c-3/2-2\beta} K_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) dy \right\}.$$

Using the integrals (B.15) and (B.16), we calculate the integrals:

$$\int_x^\infty y^{c-3/2-2\beta} K_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) dy = \left(\frac{a^2|\beta|}{2}\right) \frac{\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}}{(s+b)^{1+\frac{2c-1}{4|\beta|}}} \Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)$$

$$+ \left(\sqrt{2}a|\beta|\right)^{\nu} x^{-2\beta-1}(s+b)^{-\nu/2} \frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)} {}_1F_2\left(\begin{matrix} 1-\frac{1}{2|\beta|} & \\ 1-\nu, & 2-\frac{1}{2|\beta|}\end{matrix} ; \frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)$$

$$+ \left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2} \frac{\Gamma(-\nu)\Gamma\left(-\frac{c}{|\beta|}-1\right)}{4|\beta|\Gamma\left(-\frac{c}{|\beta|}\right)} {}_1F_2\left(\begin{matrix} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{matrix} ; \frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right),$$

$$\int_x^\infty y^{c-3/2-2\beta} I_\nu\left(\frac{y^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) dy$$

$$= \frac{\left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2}}{2(c+|\beta|)\Gamma(\nu+1)} {}_1F_2\left(\begin{matrix} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{matrix} ; \frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right).$$

Using the identity

$$K_\nu(x) = \pi\left(I_{-\nu}(x) - I_\nu(x)\right)/(2\sin(\nu\pi))$$

together with $\pi/\sin(\pi\nu) = -\Gamma(-\nu)\Gamma(1+\nu)$ and $\Gamma(-(a+1))/\Gamma(-a) = -1/(a+1)$, we obtain the resolvent $(\mathcal{R}_s 1)(x)$:

$$(\mathcal{R}_s 1)(x) = \frac{2}{a^2|\beta|} x^{\frac{1}{2}-c}$$

$$\times \left\{ -I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right) \frac{\left(\sqrt{2}a|\beta|\right)^{-\nu} x^{2c-2\beta}(s+b)^{\nu/2}\Gamma(-\nu)}{4(c+|\beta|)} {}_1F_2\left(\begin{matrix} \frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{matrix} ; \frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)\right.$$

$$+I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)\left[\left(\frac{a^2|\beta|}{2}\right)\frac{\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)+\right.$$

$$\left.\left.+\left(\sqrt{2}a|\beta|\right)^{\nu}x^{-2\beta-1}(s+b)^{-\nu/2}\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)}\,{}_1F_2\left(\begin{array}{cc}1-\frac{1}{2|\beta|} & \\ 1-\nu, & 2-\frac{1}{2|\beta|}\end{array};\frac{x^{-2\beta}(s+b)}{\left(\sqrt{2}a|\beta|\right)^2}\right)\right]\right\}.$$

The resolvent $(\mathcal{R}_s 1)(x)$ has no poles except possibly at $s=-b$ when $2|\beta|>1$ due to the second term in the parenthesis:

$$\frac{I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}\approx\left(\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu}\frac{(s+b)^{\nu/2}}{\Gamma(\nu+1)}+\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu+2}\frac{(s+b)^{\nu/2+1}}{\Gamma(\nu+2)}\right)(s+b)^{-\left(1+\frac{2c-1}{4|\beta|}\right)}$$

$$=\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu}\frac{(s+b)^{\frac{1}{2|\beta|}-1}}{\Gamma(\nu+1)}+\left(\frac{\sqrt{2}x^{-\beta}}{2a|\beta|}\right)^{\nu+2}\frac{(s+b)^{\frac{1}{2|\beta|}}}{\Gamma(\nu+2)}.$$

However, this term does not contribute to the Laplace inversion since the corresponding residue vanishes:

$$\lim_{s\to-b}(s+b)\frac{I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right)}{(s+b)^{1+\frac{2c-1}{4|\beta|}}}=0.$$

Thus, the only singularity of the resolvent is a branching point at $s=-b$. We place the branch cut along the negative real line from $-\infty$ to $-b$. Similar to part (ii) of the proof of Theorem 8.2, we use the Cauchy's Theorem to calculate the Laplace inversion integral. We consider a closed contour in Figure 1. Since the function is analytic inside the contour, the integral along the closed contour vanishes. On the other hand, the integral is equal to the sum of the integral along the line parallel to the imaginary axes in the Bromwich Laplace inversion integral, the integrals along the two arcs at infinity, the integrals along each side of the branch cut, and the integral along the arc around the branching point $s=-b$. Considering the asymptotics of the resolvent, it can be shown that the integrals along the arcs at infinity and along the arc around the branching point vanish. The analysis is similar to the one in the proof of Theorem 8.2(ii) and we omit it to save space.

Applying the Cauchy Theorem as in part (ii) of the proof of Theorem 8.2, the Laplace inversion integral reduces to the integral of the jump of the function across the branch cut (after a change of variable $s=\left(\lambda e^{\pm\pi i}-b\right)$):

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}(\mathcal{R}_s 1)(x)ds=\frac{1}{2\pi i}\int_0^\infty e^{-(\lambda+b)t}\left((\mathcal{R}_{\lambda e^{-\pi i}-b}1)(x)-(\mathcal{R}_{\lambda e^{\pi i}-b}1)(x)\right)d\lambda.$$

More explicitly, it is given by:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}(\mathcal{R}_s 1)(x)ds=\frac{1}{a^2|\beta|\pi i}x^{\frac{1}{2}-c}$$

$$\times\left\{-\frac{\left(\sqrt{2}a|\beta|\right)^{-\nu}x^{2c-2\beta}\Gamma(-\nu)}{4(c+|\beta|)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{\nu/2}\,{}_1F_2\left(\begin{array}{cc}\frac{c}{|\beta|}+1 & \\ 1+\nu, & \frac{c}{|\beta|}+2\end{array};-\frac{x^{-2\beta}\lambda}{\left(\sqrt{2}a|\beta|\right)^2}\right)\times\right.$$

$$\times\left[e^{-\frac{\nu\pi}{2}i}I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right)-e^{\frac{\nu\pi}{2}i}I_{-\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda$$

$$+\left(\frac{a^2|\beta|}{2}\right)\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)}$$

$$\times\left[e^{\left(1+\frac{2c-1}{4|\beta|}\right)\pi i}I_{\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right)-e^{-\left(1+\frac{2c-1}{4|\beta|}\right)\pi i}I_{\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda$$

$$+\left(\sqrt{2}a|\beta|\right)^{\nu}x^{-2\beta-1}\frac{\Gamma(\nu)\Gamma\left(\frac{1}{2|\beta|}-1\right)}{4|\beta|\Gamma\left(\frac{1}{2|\beta|}\right)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\nu/2}\,{}_1F_2\left(\begin{matrix}1-\frac{1}{2|\beta|} & \\ 1-\nu, & 2-\frac{1}{2|\beta|}\end{matrix};-\frac{x^{-2\beta}\lambda}{\left(\sqrt{2}a|\beta|\right)^2}\right)\times$$

$$\times\left[e^{\frac{\nu\pi}{2}i}I_{\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{-\frac{\pi}{2}i}}{a|\beta|}\right)-e^{-\frac{\nu\pi}{2}i}I_{\nu}\left(\frac{x^{-\beta}\sqrt{2\lambda}e^{\frac{\pi}{2}i}}{a|\beta|}\right)\right]d\lambda\Bigg\}.$$

Using the property

$$I_\nu\left(e^{\pm\frac{\pi}{2}i}a\right)=e^{\pm i\frac{\nu\pi}{2}}J_\nu(a),$$

we can verify that the first and third integrals vanish, and the remaining integral takes the form:

$$\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}(\mathcal{R}_s1)(x)ds=x^{\frac{1}{2}-c}\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}\Gamma\left(1-\frac{1}{2|\beta|}\right)\Gamma\left(\frac{c}{|\beta|}+1\right)$$

$$\times\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)}J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)\left[\frac{e^{-\left(\frac{1}{2|\beta|}-1\right)\pi i}-e^{\left(\frac{1}{2|\beta|}-1\right)\pi i}}{2\pi i}\right]d\lambda.$$

Finally, using the identity $\pi/\sin(\pi\nu)=-\Gamma(-\nu)\Gamma(1+\nu)$, we obtain:

$$\mathbb{Q}(\zeta>t)=\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}e^{st}(\mathcal{R}_s1)(x)ds$$

$$=x^{\frac{1}{2}-c}\left(\sqrt{2}a|\beta|\right)^{\frac{2c-1}{2|\beta|}}\frac{\Gamma\left(\frac{c}{|\beta|}+1\right)}{\Gamma\left(\frac{1}{2|\beta|}\right)}\int_0^\infty e^{-(\lambda+b)t}\lambda^{-\left(1+\frac{2c-1}{4|\beta|}\right)}J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)d\lambda,$$

which completes the proof. □

A.4 Proof of Theorem 8.4

Since the payoff $f(x)=(k-x)^+$ is in the Hilbert space $L^2((0,\infty),\mathfrak{m})$, we can apply the spectral expansion approach as described in section 6. (i) When $\mu+b>0$, by Theorem 8.2(i) the spectrum is purely discrete with eigenvalues and eigenfunctions (8.11). We need to compute the eigenfunction expansion coefficients:

$$c_n=\int_0^k(k-y)\,\varphi_n(y)\mathfrak{m}(y)dy$$

$$= \frac{2A^{\frac{\nu}{2}}}{a^2}\sqrt{\frac{(n-1)!(\mu+b)}{\Gamma(\nu+n)}}\left(k\int_0^k y^{2c-2\beta-1}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy - \int_0^k y^{2c-2\beta}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy\right).$$

Using the integral (B.18), we obtain:

$$\int_0^k y^{2c-2\beta-1}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy = \frac{k^{2c-2\beta}\left(1+\nu\right)_{n-1}}{2\left(c+|\beta|\right)(n-1)!}\,{}_2F_2\left(\begin{array}{cc} 1-n, & 1+\frac{c}{|\beta|} \\ 1+\nu, & 2+\frac{c}{|\beta|}\end{array};Ak^{-2\beta}\right),$$

and

$$\int_0^k y^{2c-2\beta}L_{n-1}^{(\nu)}\left(Ay^{-2\beta}\right)dy = \frac{k^{2c+1-2\beta}\left(1+\nu\right)_{n-1}}{\left(2c+1+2|\beta|\right)(n-1)!}\,{}_1F_1\left(1-n;2+\nu;Ak^{-2\beta}\right).$$

Finally, using the identities ${}_1F_1\left(a,b;z\right) = \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}L_{-a}^{b-1}(z)$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we obtain the expression (8.15).

(ii) For $\mu+b=0$ we need to compute the integral:

$$c\left(\lambda\right) = \frac{1}{a^2|\beta|}\left(k\int_0^k y^{c-\frac{3}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy - \int_0^k y^{c-\frac{1}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy\right).$$

Using the integral (B.14), we obtain:

$$\int_0^k y^{c-\frac{3}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy = \frac{k^{2c-2\beta}}{2\left(c+|\beta|\right)\Gamma\left(1+\nu\right)}\left(\frac{\sqrt{\lambda}}{\sqrt{2}a|\beta|}\right)^\nu {}_1F_2\left(\begin{array}{cc} 1+\frac{c}{|\beta|} & \\ 2+\frac{c}{|\beta|}, & 1+\nu\end{array};-\frac{k^{-2\beta}\lambda}{2\left(a|\beta|\right)^2}\right),$$

and

$$\int_0^k y^{c-\frac{1}{2}-2\beta}J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right)dy = \frac{k^{2c+1-2\beta}}{\left(2c+1+2|\beta|\right)\Gamma\left(1+\nu\right)}\left(\frac{\sqrt{\lambda}}{\sqrt{2}a|\beta|}\right)^\nu {}_0F_1\left(;2+\nu;-\frac{k^{-2\beta}\lambda}{2\left(a|\beta|\right)^2}\right).$$

Substituting these into the integral for $c(\lambda)$, we obtain:

$$c\left(\lambda\right) = \frac{k^{2c+1-2\beta}\lambda^{\nu/2}}{\left(\sqrt{2}a|\beta|\right)^{\nu+2}\Gamma\left(1+\nu\right)}$$

$$\times\left(\frac{|\beta|}{\left(c+|\beta|\right)}\,{}_1F_2\left(\begin{array}{cc} 1+\frac{c}{|\beta|} & \\ 2+\frac{c}{|\beta|}, & 1+\nu\end{array};-\frac{k^{-2\beta}\lambda}{2\left(a|\beta|\right)^2}\right) - \frac{1}{(\nu+1)}\,{}_0F_1\left(;2+\nu;-\frac{k^{-2\beta}\lambda}{2\left(a|\beta|\right)^2}\right)\right).$$

Using the identity

$${}_0F_1\left(;b;z\right) = \left(-z\right)^{\frac{1-b}{2}}\Gamma(b)J_{b-1}\left(2\sqrt{-z}\right),$$

we finally obtain the explicit expression (8.17). □

B Special Functions

This Appendix collects some facts about special functions appearing in the solution of the time changed JDCEV model in this paper. The reader is referred to Abramowitz and Stegun (1972), Buchholz (1969), Gradshteyn and Ryzhik (1994), Prudnikov et al. (1986), (1990) and Slater (1960) for further details. All the special functions in this Appendix are available as built-in functions in *Mathematica* and *Maple* software packages. To compute these functions efficiently, these packages use a variety of integral and asymptotic representations given in the above references in addition to the defining hypergeometric series presented here.

B.1 Hypergeometric Functions

The generalized hypergeometric function is defined by the generalized hypergeometric series:

$$ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z) \equiv {}_pF_q\left(\begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}, \tag{B.1} $$

where $(a)_n = a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol (and $\Gamma(z)$ is the Gamma function). The regularized function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)/(\Gamma(b_1)...\Gamma(b_q))$ is analytic for all values of $a_1, ..., a_p, b_1, ..., b_q$, and z real or complex. The most well-known special cases are the Gauss hypergeometric function ${}_2F_1(a_1, a_2; b; z)$ and the Kummer confluent hypergeometric function ${}_1F_1(a; b; z)$.

The second confluent hypergeometric function (Tricomi function) is defined in terms of the Kummer function:

$$ U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left\{ \frac{{}_1F_1(a; b; z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{z^{1-b}\, {}_1F_1(1+a-b; 2-b; z)}{\Gamma(a)\Gamma(2-b)} \right\}. \tag{B.2} $$

It is analytic for all values of a, b, and z real or complex even when b is zero or a negative integer, for in these cases it can be defined in the limit $b \to \pm n$ or 0. The confluent hypergeometric functions are solutions of the confluent hypergeometric equation:

$$ z\frac{d^2u}{dz^2} + (b-z)\frac{du}{dz} - au = 0. \tag{B.3} $$

B.2 Whittaker Functions

The Whittaker functions arise as solutions to the Whittaker equation:

$$ \frac{d^2w}{dz^2}(z) + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right) w(z) = 0. \tag{B.4} $$

They can be expressed in terms of the confluent hypergeometric functions:

$$ M_{k,m}(z) = e^{-z/2} z^{m+1/2}\, {}_1F_1(1/2 + m - k; 2m + 1; z) \tag{B.5} $$

$$ W_{k,m}(z) = e^{-z/2} z^{m+1/2} U(1/2 + m - k, 2m + 1, z) \tag{B.6} $$

In turn the confluent hypergeometric functions can be expressed in terms of the Whittaker functions:

$$ {}_1F_1(a; b; \pm z) = z^{-b/2} e^{\pm z/2} M_{b/2-a,(b-1)/2}(z) \tag{B.7} $$

$$ U(a, b, z) = z^{-b/2} e^{z/2} W_{b/2-a,(b-1)/2}(z) \tag{B.8} $$

Due to (B.2), the Whittaker function W can be expressed in terms of the function M:

$$ W_{k,m}(x) = \frac{\pi}{\sin(2m\pi)} \left[\frac{M_{k,-m}(x)}{\Gamma(1/2 + m - k)\Gamma(1 - 2m)} - \frac{M_{k,m}(x)}{\Gamma(1/2 - m - k)\Gamma(1 + 2m)} \right]. \tag{B.9} $$

For details on the Whittaker functions and their properties see Slater (1960) and Buchholz (1969).

The Whittaker function $M_{k,m}(x)$ satisfies the following multiplication identity (Slater (1960), Eq.(2.6.18), p.30):

$$M_{k,m}(xy) = e^{\frac{1}{2}x(y-1)} y^{-k} \sum_{n=0}^{\infty} \frac{(y-1)^n}{n! y^n} (1/2+m+k)_n \, M_{k+n,m}(x) \tag{B.10}$$

Using the multiplication theorem and the connection between the Whittaker and Kummer functions (B.5), one can show that the eigenfunction expansion for the survival probability (8.13) collapses to the closed-form expression (5.14) obtained in Carr and Linetsky (2006).

B.3 Integrals with Special Functions

In this subsection we collect a number of integrals necessary for the proofs of Theorems 8.2–8.4.

B.3.1 Integrals with Whittaker functions.

Prudnikov et al. (1990), Eq.(1.13.1.1), p.39:

$$\int_0^x x^{\alpha-1} e^{\pm\frac{a}{2}x} M_{\rho,\sigma}(ax)\,dx = \frac{2a^{\sigma+\frac{1}{2}} x^{\alpha+\sigma+\frac{1}{2}}}{2\alpha+2\sigma+1} \, {}_2F_2\left(\begin{matrix} \alpha+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ \alpha+\sigma+\frac{3}{2}, & 2\sigma+1 \end{matrix} ; \pm ax \right)$$

valid for $\Re(\alpha+\sigma+\frac{1}{2}) > 0$ and $x > 0$. Changing the integration variable $x = y^\delta$ and setting $\gamma := \alpha\delta - 1$, we obtain the following integral:

$$\int_0^y y^\gamma e^{\pm\frac{a}{2}y^\delta} M_{\rho,\sigma}\left(ay^\delta\right) dy \tag{B.11}$$

$$= \frac{2a^{\sigma+\frac{1}{2}} y^{(\gamma+1)+\delta\left(\sigma+\frac{1}{2}\right)}}{2(\gamma+1)+\delta(2\sigma+1)} \, {}_2F_2\left(\begin{matrix} \frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ \frac{\gamma+1}{\delta}+\sigma+\frac{3}{2}, & 2\sigma+1 \end{matrix} ; \pm ay^\delta \right)$$

valid for $\Re\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right) > 0$, $\delta > 0$ and $y > 0$.

Prudnikov et al. (1990), Eq.(1.13.2.1), p.40:

$$\int_x^\infty x^{\alpha-1} e^{\pm\frac{a}{2}x} W_{\rho,\sigma}(ax)\,dx = -\frac{2a^{\sigma+\frac{1}{2}} x^{\alpha+\sigma+\frac{1}{2}}}{2\alpha+2\sigma+1} \frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\rho-\sigma\right)} \, {}_2F_2\left(\begin{matrix} \alpha+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ 2\sigma+1, & \alpha+\sigma+\frac{3}{2} \end{matrix} ; \pm ax \right)$$

$$-\frac{2a^{\frac{1}{2}-\sigma} x^{\alpha-\sigma+\frac{1}{2}}}{2\alpha-2\sigma+1} \frac{\Gamma(2\sigma)}{\Gamma\left(\frac{1}{2}-\rho+\sigma\right)} \, {}_2F_2\left(\begin{matrix} \alpha-\sigma+\frac{1}{2}, & \frac{1}{2}-\sigma\mp\rho \\ 1-2\sigma, & \alpha-\sigma+\frac{3}{2} \end{matrix} ; \pm ax \right) + a^{-\alpha} A_\pm,$$

where:

$$A_+ = \frac{\Gamma\left(\alpha+\sigma+\frac{1}{2}\right)\Gamma\left(\alpha-\sigma+\frac{1}{2}\right)\Gamma(-\alpha-\rho)}{\Gamma\left(\sigma+\frac{1}{2}-\rho\right)\Gamma\left(\frac{1}{2}-\rho-\sigma\right)}, \quad A_- = \frac{\Gamma\left(\alpha+\sigma+\frac{1}{2}\right)\Gamma\left(\alpha-\sigma+\frac{1}{2}\right)}{\Gamma(\alpha-\rho+1)}.$$

The integral is valid for $\Re(a) > 0$, $\Re(\alpha+\rho) < 0$, $x > 0$, $|\arg(a)| < \frac{3\pi}{2}$. Changing the integration variable $x = y^\delta$ and setting $\gamma = \alpha\delta - 1$, we obtain the following integral:

$$\int_y^\infty y^\gamma e^{\pm\frac{a}{2}y^\delta} W_{\rho,\sigma}\left(ay^\delta\right) dy \tag{B.12}$$

$$= -\frac{2a^{\sigma+\frac{1}{2}}y^{\gamma+1+\delta\left(\sigma+\frac{1}{2}\right)}}{2(\gamma+1)+\delta(2\sigma+1)} \times \frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\rho-\sigma\right)}\, {}_2F_2\left(\begin{matrix} \frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}, & \sigma+\frac{1}{2}\mp\rho \\ 2\sigma+1, & \frac{\gamma+1}{\delta}+\sigma+\frac{3}{2}\end{matrix};\pm a y^{\delta}\right)$$

$$-\frac{2a^{\frac{1}{2}-\sigma}y^{\gamma+1-\delta\left(\sigma-\frac{1}{2}\right)}}{2(\gamma+1)-\delta(2\sigma-1)}\frac{\Gamma(2\sigma)}{\Gamma\left(\frac{1}{2}-\rho+\sigma\right)}\, {}_2F_2\left(\begin{matrix} \frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}, & \frac{1}{2}-\sigma\mp\rho \\ 1-2\sigma, & \frac{\gamma+1}{\delta}-\sigma+\frac{3}{2}\end{matrix};\pm a y^{\delta}\right)+\frac{a^{-\frac{\gamma+1}{\delta}}}{\delta}A_{\pm},$$

where:

$$A_{+} = \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right)\Gamma\left(\frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}\right)\Gamma\left(-\frac{\gamma+1}{\delta}-\rho\right)}{\Gamma\left(\sigma+\frac{1}{2}-\rho\right)\Gamma\left(\frac{1}{2}-\rho-\sigma\right)}$$

$$A_{-} = \frac{\Gamma\left(\frac{\gamma+1}{\delta}+\sigma+\frac{1}{2}\right)\Gamma\left(\frac{\gamma+1}{\delta}-\sigma+\frac{1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}-\rho+1\right)}.$$

The integral (B.12) is valid for $\Re(a) > 0,\ \delta > 0,\ \Re\left(\frac{\gamma+1}{\delta}+\rho\right) < 0,\ y > 0,\ |\arg(a)| < \frac{3\pi}{2}$.

B.3.2 Integrals with Bessel functions.

Gradshteyn and Ryzhik (2000), Eq.(6.643.1), p.701:

$$\int_0^{\infty} e^{-\alpha x}x^{\mu-1/2}J_{2\nu}\left(2\beta\sqrt{x}\right)dx = \frac{\Gamma(\mu+\nu+1/2)}{\beta\Gamma(2\nu+1)}e^{-\frac{\beta^2}{2\alpha}}\alpha^{-\mu}M_{\mu,\nu}\left(\frac{\beta^2}{\alpha}\right). \tag{B.13}$$

Prudnikov et al. (1988), Eq.(2.12.3.1), p.175 (set $\beta = 1$):

$$\int_0^{a} x^{\alpha-1}J_{\nu}(cx)dx = \left(\frac{c}{2}\right)^{\nu}\frac{a^{\alpha+\nu}}{\Gamma(\nu+1)(\alpha+\nu)}\, {}_1F_2\left(\begin{matrix} \frac{\alpha+\nu}{2}, & \\ \nu+1, & \frac{\alpha+\nu}{2}+1\end{matrix};-\frac{a^2c^2}{4}\right)$$

valid for $a > 0,\ \Re(\alpha+\nu) > 0$. Introducing a new integration variable $x = y^{\delta}$ with $\delta > 0$ and setting $a = b^{\delta}$, we obtain the following integral:

$$\int_0^{b} y^{\gamma}J_{\nu}(cy^{\delta})dy = \left(\frac{c}{2}\right)^{\nu}\frac{b^{\gamma+1+\delta\nu}}{\Gamma(\nu+1)(\gamma+1+\delta\nu)}\, {}_1F_2\left(\begin{matrix} \frac{\gamma+1+\delta\nu}{2\delta}, & \\ \nu+1, & \frac{\gamma+1+\delta\nu}{2\delta}+1\end{matrix};-\frac{b^{2\delta}c^2}{4}\right) \tag{B.14}$$

valid for $b^{\delta} > 0,\ \Re(\gamma+1+\delta\nu) > 0$.

Prudnikov et al. (1988), Eq.(2.15.2.5), p.302 (set $\beta = 1$):

$$\int_0^{a} x^{\alpha-1}I_{\nu}(cx)dx = 2^{-\nu-1}a^{\alpha+\nu}c^{\nu}\frac{\Gamma\left(\frac{\alpha+\nu}{2}\right)}{\Gamma(\nu+1)\Gamma\left(1+\frac{\alpha+\nu}{2}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\nu+\alpha}{2}, & \\ 1+\nu, & 1+\frac{\nu+\alpha}{2}\end{matrix};\frac{(ac)^2}{4}\right)$$

valid for $a > 0,\ \Re(\alpha+\nu) > 0$. Introducing a new integration variable $x = y^{\delta}$ with $\delta > 0$ and setting $a = b^{\delta}$, we obtain the following integral:

$$\int_0^{b} y^{\gamma}I_{\nu}(cy^{\delta})dy = \frac{b^{\gamma+1+\delta\nu}c^{\nu}}{2^{\nu+1}\delta}\frac{\Gamma\left(\frac{\gamma+1+\delta\nu}{2\delta}\right)}{\Gamma(\nu+1)\Gamma\left(1+\frac{\gamma+1+\delta\nu}{2\delta}\right)}\, {}_1F_2\left(\begin{matrix} \frac{\gamma+1+\delta\nu}{2\delta}, & \\ 1+\nu, & 1+\frac{\gamma+1+\delta\nu}{2\delta}\end{matrix};\frac{c^2b^{2\delta}}{4}\right) \tag{B.15}$$

valid for $b^{\delta} > 0,\ \Re(\gamma+1+\delta\nu) > 0$.

Prudnikov et al. (1988), Eq.(2.16.3.7), p.345 (set $\beta = 1$):

$$\int_a^\infty x^{\alpha-1}K_\nu(cx)dx = 2^{\nu-2}a^{\alpha-\nu}c^{-\nu}\Gamma(\nu)\frac{\Gamma\left(\frac{\nu-\alpha}{2}\right)}{\Gamma\left(1+\frac{\nu-\alpha}{2}\right)}\,{}_1F_2\left(\begin{matrix}\frac{\alpha-\nu}{2}, & \\ 1-\nu, & 1+\frac{\alpha-\nu}{2}\end{matrix};\frac{(ac)^2}{4}\right)$$

$$+2^{-\nu-2}a^{\alpha+\nu}c^{\nu}\Gamma(-\nu)\frac{\Gamma\left(-\frac{\nu+\alpha}{2}\right)}{\Gamma\left(1-\frac{\nu+\alpha}{2}\right)}\,{}_1F_2\left(\begin{matrix}\frac{\nu+\alpha}{2}, & \\ 1+\nu, & 1+\frac{\nu+\alpha}{2}\end{matrix};\frac{(ac)^2}{4}\right)+2^{\alpha-2}c^{-\alpha}\Gamma\left(\frac{\nu+\alpha}{2}\right)\Gamma\left(\frac{\alpha-\nu}{2}\right)$$

valid for $a > 0,\ \Re(c) > 0$. Introducing a new integration variable $x = y^\delta$ with $\delta > 0$ and setting $a = b^\delta$, we obtain the following integral:

$$\int_b^\infty y^\gamma K_\nu(cy^\delta)dy \tag{B.16}$$

$$= \frac{1}{\delta}2^{\nu-2}b^{\gamma+1-\delta\nu}c^{-\nu}\Gamma(\nu)\frac{\Gamma\left(\frac{\delta\nu-(\gamma+1)}{2\delta}\right)}{\Gamma\left(1+\frac{\delta\nu-(\gamma+1)}{2\delta}\right)}\,{}_1F_2\left(\begin{matrix}\frac{(\gamma+1)-\delta\nu}{2\delta}, & \\ 1-\nu, & 1+\frac{(\gamma+1)-\delta\nu}{2\delta}\end{matrix};\frac{c^2b^{2\delta}}{4}\right)$$

$$+\frac{1}{\delta}2^{-\nu-2}b^{\gamma+1+\delta\nu}c^{\nu}\Gamma(-\nu)\frac{\Gamma\left(-\frac{\nu\delta+\gamma+1}{2\delta}\right)}{\Gamma\left(1-\frac{\nu\delta+\gamma+1}{2\delta}\right)}\,{}_1F_2\left(\begin{matrix}\frac{\nu\delta+\gamma+1}{2\delta}, & \\ 1+\nu, & 1+\frac{\nu\delta+\gamma+1}{2\delta}\end{matrix};\frac{c^2b^{2\delta}}{4}\right)$$

$$+\frac{1}{\delta}2^{\frac{\gamma+1}{\delta}-2}c^{-\frac{\gamma+1}{\delta}}\Gamma\left(\frac{\nu\delta+\gamma+1}{2\delta}\right)\Gamma\left(\frac{(\gamma+1)-\delta\nu}{2\delta}\right)$$

valid for $b^\delta > 0,\ \Re(c) > 0$.

B.3.3 Integrals with Generalized Laguerre Polynomials.

Prudnikov et al. (1988), Eq.(1.14.3.3), p.51:

$$\int_{x_1}^{x_2} x^\lambda L_n^{(\alpha)}(ax)dx = \pm\frac{(\alpha+1)_n}{n!(\lambda+1)}x^{\lambda+1}\,{}_2F_2\left(\begin{matrix}-n, & \lambda+1\\ \alpha+1, & \lambda+2\end{matrix};ax\right), \tag{B.17}$$

where $x_1 = 0$, $x_2 = x$ and $\Re(\lambda) > -1$ (plus sign) or $x_1 = x$, $x_2 = \infty$ and $\Re(\lambda) < -n-1$ (minus sign). Introducing a new integration variable $x = y^\delta$ with $\delta > 0$ and setting $x_1 = b_1^\delta$, $x_2 = b_2^\delta$, and $\gamma = (\lambda+1)\delta - 1$, we obtain:

$$\int_{b_1}^{b_2} y^\gamma L_n^{(\alpha)}(ay^\delta)dy = \pm\frac{(\alpha+1)_n}{n!\,(\gamma+1)}b^{\gamma+1}\,{}_2F_2\left(\begin{matrix}-n, & \frac{\gamma+1}{\delta}\\ \alpha+1, & \frac{\gamma+1}{\delta}+1\end{matrix};ab^\delta\right) \tag{B.18}$$

where $b_1 = 0$, $b_2 = b$ and $\Re\left(\frac{\gamma+1}{\delta}\right) > 0$ (plus sign in (B.17)) or $b_1 = b$, $b_2 = \infty$ and $\Re\left(\frac{\gamma+1}{\delta}\right) < -n$ (minus sign in (B.17)).

C Examples of Subordinators

A simple example of a finite activity subordinator is a compound Poisson process with jump arrival rate $\alpha > 0$ and exponentially distributed jumps with mean $1/\eta > 0$ with the Lévy measure: $\nu(ds) = \alpha\eta e^{-\eta s}ds$. The Laplace exponent (3.2) of a subordinator with this Lévy measure and drift $\gamma > 0$ is: $\phi(\lambda) = \gamma\lambda + \frac{\alpha\lambda}{\lambda+\eta}$ and $\mathcal{I}_\nu = (-\infty, \eta)$.

A more general compound Poisson Lévy measure is of the form $\nu(ds) = \alpha F(ds)$, where $F(ds)$ is a probability measure on $\mathbb{R}^+$ so that positive jumps arrive according to a Poisson process with intensity α and are distributed according to F. The Laplace exponent (3.2) for the compound Poisson subordinator simplifies to: $\phi(\lambda) = \gamma\lambda + \alpha[1 - \mathcal{L}(F)(\lambda)]$, where $\mathcal{L}(F)(s)$ is the Laplace transform of the probability measure F.

An important family of Lévy subordinators is defined by the following three-parameter family of Lévy measures

$$\nu(ds) = Cs^{-Y-1}e^{-\eta s}ds$$

with $C > 0$, $\eta > 0$, and $Y < 1$. For $Y \in (0, 1)$ these are the so-called *tempered stable subordinators* (exponentially dampened counterparts of stable subordinators with $\nu(ds) = Cs^{-Y-1}ds$). The special case $Y = 1/2$ is known as the *inverse Gaussian process* (Barndorff-Nielsen (1998)). The limiting case $Y = 0$ is the *gamma process* (see Madan et al. (1998)). The processes with $Y \in [0, 1)$ are infinite activity processes. For $Y < 0$ these are compound Poisson processes with gamma distributed jump sizes. The previously discussed compound Poisson process with exponential jumps is a special case with $Y = -1$ (and $C = \alpha\eta$). For $Y \neq 0$ the Laplace exponent (3.2) is given by: $\phi(\lambda) = \gamma\lambda - C\Gamma(-Y)[(\lambda + \eta)^Y - \eta^Y]$, where $\Gamma(x)$ is the gamma function. For the gamma process with $Y = 0$ and drift $\gamma \geq 0$ the Laplace exponent (3.2) is given by: $\phi(\lambda) = \gamma\lambda + C\ln(1 + \lambda/\eta)$. For $Y \in [0, 1)$ the transition measures $\pi_t(ds)$ are known in closed form only for the two special cases with $Y = 0$ (gamma process) and $Y = 1/2$ (inverse Gaussian process) and are given by:

$$\pi_t^G(ds) = \frac{\eta^{Ct}}{\Gamma(Ct)}(s + \gamma t)^{Ct-1}e^{-\eta(s+\gamma t)}ds,$$

$$\pi_t^{IG}(ds) = \frac{Ct}{(s + \gamma t)^{3/2}} \exp\left(2Ct\sqrt{\pi\eta} - \eta(s + \gamma t) - \pi C^2t^2/(s + \gamma t)\right) ds,$$

respectively. For $Y < 0$, the transition measure of the compound Poisson process (CPP) with gamma distributed jumps is (the CPP with exponential jumps discussed above is a special case with $Y = -1$):

$$\pi_t(ds) = e^{-\alpha t}\delta_{\{\gamma t\}}(ds) + \sum_{n=1}^{\infty} e^{-\alpha t}\frac{(\alpha t\eta^{|Y|})^n}{n!\Gamma(n|Y|)}(s + \gamma t)^{n|Y|-1}e^{-\eta(s+\gamma t)}ds,$$

where $\delta_{\{\gamma t\}}(ds)$ is the Dirac measure with unit mass at $s = \gamma t$ ($e^{-\alpha t}$ is the probability of no jumps by time t). The interval $\mathcal{I}_\nu = (-\infty, \eta]$ for $Y \in (0, 1)$ and $\mathcal{I}_\nu = (-\infty, \eta)$ for $Y \leq 0$. For general $Y \in (0, 1)$ the transition measure is not known in closed form and has to be computed numerically by inverting the Laplace transform (3.4).

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