Unified Credit-Equity Modeling*

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1 Introduction

In the celebrated Black-Scholes-Merton options pricing model the firm's stock price is assumed to follow geometric Brownian motion, a strictly positive diffusion process with infinite lifetime. This assumption precludes the possibility of corporate bankruptcy rendering equity worthless. In contrast, the literature on credit risk is concerned with modeling bankruptcy and credit spreads. Until relatively recently, there was a disconnect between equity options pricing models that focus on modeling implied volatility smiles and ignore the possibility of default and models used to value corporate debt and credit derivatives that focus on default exclusively and ignore the information available in the equity options market.

As equity derivatives markets and credit derivatives markets have developed over the past decade, it has become increasingly clear to market participants that there are close linkages between credit derivatives, such as credit default swaps (CDS), and equity options. It has been repeatedly observed in the markets that sharp stock price drops and accompanying increases in implied volatilities of stock options tend to happen contemporaneously with sharp increases in market credit spreads on corporate debt and credit default swaps. In particular, deep out-of-the-money puts derive most of their value from the positive probability of bankruptcy that will render equity worthless. Time and again, we have seen that when credit market participants become particularly concerned about default of a firm on its debt, transaction volumes in both the firm's CDS and deep out-of-the-money puts on the firm's stock sharply increase. Moreover, implied volatility values extracted from deep out-of-the-money puts sharply increase to very high values. This market behavior is not surprising, as deep out-of-the-money puts can be used to hedge against corporate default, thus effectively blurring the line between credit derivatives and equity derivatives (see Carr and Wu (2009) for an empirical investigation of the connection between CDS and deep out-of-the-money puts). In fact, this line is already blurred when one thinks of hybrid corporate securities such as convertible bonds that have both equity and credit

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features. This line is further blurred with the recent introduction of new hybrid derivatives, such as equity default swaps (EDS).

Among recent empirical studies documenting linkages between equity volatility and credit spreads, Campbell and Taksler (2003) show that firm-level volatility can explain as much cross-sectional variation in bond yields as can credit ratings. Cremers et al. (2008) shows that CDS spreads are positively correlated with both stock option implied volatility levels and with the (negative) slope of the implied volatility as a function of moneyness. Hilscher (2008), using Merton's (1974) model, calculates a measure of implied volatility from corporate bond yield spreads and finds that this measure is highly significant when forecasting volatility. Carr and Wu (2009) show that stock return volatility and credit spreads have close links, specially at short maturities, due to positive co-movements between the diffusion variance rate and the default arrival rate. Zhang et al. (2009) also examine the relationship between stock volatility and credit spreads finding that equity volatility and a jump risk measure together explain 75% of the variation in credit spreads.

In light of this, a new generation of hybrid credit-equity models are needed that allow one to value and risk manage all securities related to a given firm, including equity and credit derivatives, in a unified fashion. On one hand, one might contemplate development of a comprehensive structural model that explicitly models the capital structure of a firm with its debt and equity as contingent claims on the value of the assets of the firm. However, such models would be exceedingly difficult to implement and calibrate in practice, as equity derivatives would be treated as compound options on the unobserved value of the assets of the firm. Practical considerations necessitate one to turn to reduced-form models instead.

The pre-cursor of such models is Merton's (1976) model with a jump to default, where the stock of a firm evolves according to geometric Brownian motion punctuated with a single jump that takes the stock price from a positive value to zero (default or bankruptcy state). In the original Merton's model the jump to default arrives according to a Poisson process with constant arrival rate (default intensity) independent of the firm's stock price. This is clearly not empirically realistic, as one would expect the probability of a jump to default to increase at lower stock prices and decrease at higher stock prices. It is thus natural to take the default intensity to be a decreasing function of the stock price. Indeed, it is in the convertible bonds pricing literature where such a specification for the default intensity first emerged. Davis and Lischka (2002), Andersen and Buffum (2003/2004) and Ayache et al. (2003) have all specified the default intensity as the negative power of the stock price. These references have treated the model numerically, e.g., by solving the pricing PDE by numerical methods. This specification of the default intensity as the negative power of the stock price has become quite popular for the pricing of convertible bonds and other hybrid products (e.g., Das and Sundaram (2007)).

In contrast to the numerical approaches in the literature, recently Linetsky (2006) has solved the negative power intensity model in closed form, obtaining explicit analytical solutions for both corporate bonds and stock options in this model. Analysis of these solutions makes it clear that introducing a default intensity specified as the negative power of the stock price induces implied volatility skews in the stock options prices, with the *steepness of the skew controlled by the parameters of the default intensity*. We call this model *jump-to-default extended Black-Scholes*. While this model establish a link between the implied volatility of equity options and the probability of default, since the local diffusion volatility of the stock price process remains constant in this model, the probability of default explains *all* of the volatility skew in this model.

Carr and Linetsky (2006) relaxed the assumption of constant diffusion volatility and replaced it with the constant elasticity of variance (CEV) assumption, thus introducing a *jump-to-default extended CEV model (JDCEV)*. The stock price follows a diffusion process with the CEV local diffusion volatility punctuated by a jump to default (jump to zero that renders the stock worthless). The arrival rate of default is specified to be an affine function of the local CEV variance. Since the CEV volatility is a negative power of the stock price, the JDCEV default intensity is also a negative power of the stock price (plus a constant or a deterministic function of time independent of the stock price). Carr and Linetsky solved the JDCEV model in closed form by exploiting a close connection of the CEV process to Bessel processes. Part of the implied volatility skew in the JDCEV options prices is explained by the leverage effect in the CEV volatility controlled by the CEV elasticity parameter, while part of the skew is explained by the probability of default controlled by the parameters in the default intensity specification. Vidal-Nunes (2009) has derived analytical approximations for American-style options in the JDCEV model. Le (2007) has conducted an empirical investigation of the JDCEV model. The JDCEV model has recently been applied to the pricing of equity default swaps (EDS) by Mendoza-Arriaga and Linetsky (2009) who solve the first passage time problem for the JDCEV process through a lower positive barrier and investigate the term structure of EDS spreads as a function of the EDS triggering barrier and recover the JDCEV CDS spreads in the limiting case of the default barrier at the zero stock value. Related literature on EDS and other hybrid credit-equity derivatives includes Albanese and Chen (2005), Atlan and Leblanc (2005, 2006), Campi and Sbuelz (2005), and Campi et al. (2009).

Carr and Linetsky (2006) also present a general framework for *jump-to-default extended diffusion* models with general local volatility $\sigma(S,t)$ and default intensity $\lambda(S,t)$ functions. This framework introduces the possibility of bankruptcy in the one-dimensional diffusion model with local volatility taken to be a function of the stock price and time. Carr and Madan (2010) conduct further mathematical and empirical analysis of this modeling framework. In particular, they estimate the model jointly on CDS and equity options data. Bielecki et al. (2009) study convertible bonds and more general convertible securities in the jump-to-default extended diffusion framework.

Limitations of the jump-to-default extended diffusion framework are the absence of jumps in the stock price other than the single jump to default and the absence of an independent stochastic volatility component. A class of jump-to-default extended stochastic volatility models introduce stochastic volatility both into the instantaneous diffusion volatility, as well as into the default intensity. Carr and Wu (2008a,b) propose and empirically test an affine model that is a jump-to-default extension of Heston's stochastic volatility model. The stock price follows an affine process with CIR stochastic volatility correlated with the stock price process punctuated by a jump to default. The default intensity is taken to be an affine function of the instantaneous stock price variance, as well as an additional default intensity factor also modeled by a CIR process. The model thus has three stochastic factors: the stock price, stochastic volatility, and the default intensity factor. Since the model is in the affine class, its characteristic function is known in closed form, and the pricing is accomplished by Fourier inversion. Carr and Wu estimate the model jointly on CDS and equity options data. Kovalov and Linetsky (2008) study convertible bond pricing in a four-factor extension of Carr and Wu's model that also includes stochastic interest rates. They obtain analytical solutions for European-style derivatives and develop finite element methods for numerical solution of a non-linear PDE arising in the penalty formulation of a differential game with conversion and call.

The limitations of the jump-to-default extended Heston model are the absence of jumps and the inability to introduce direct dependence of the local volatility on the stock price (such as in SABR-type models) without destroying the affine structure of the model. Mendoza-Arriaga et al. (2009) introduce a far reaching generalization of the JDCEV model that removes both of these limitations. They introduce both jumps and stochastic volatility into the JDCEV model via the mechanism of stochastic time changes and, at the same time, preserves analytical tractability. They introduce a rich class of hybrid credit-equity models with state-dependent jumps, local-stochastic volatility and default intensity. When the time change is a Lévy subordinator, the stock price process exhibits jumps with state-dependent Lévy measure. When the time change is a time integral of an activity rate process, the stock price process has local-stochastic volatility and default intensity. When the time change process is a Lévy subordinator in turn time changed with a time integral of an activity rate process, the stock price process has state-dependent jumps, local-stochastic volatility and default intensity. Mendoza-Arriaga et al. (2009) develop analytical approaches to the pricing of credit and equity derivatives in this class of models based on the Laplace transform inversion and the spectral expansion approach.

The rest of this survey is organized as follows. In Section 2 we present a survey of the jump-to-default extended diffusion modeling framework, following Carr and Linetsky (2006) and Mendoza-Arriaga and Linetsky (2009). In Section 3 we present a survey of the JDCEV model of Carr and Linetsky (2006). In Section 4 we present a survey of the time changed JDCEV model of Mendoza-Arriaga et al. (2009) with state-dependent jumps, stochastic volatility, and default.

2 Jump-to-Default Extended Diffusions (JDED)

2.1 The Modeling Framework

We start with a probability space given by $(\Omega, \mathcal{G}, \mathbb{Q})$ carrying a standard Brownian motion $\{B_t, t \geq 0\}$ and an exponential random variable $e \sim Exp(1)$ with unit parameter independent of B_t. We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{Q}$ as given. We model the *pre-default* stock dynamics under the EMM as a time-inhomogeneous diffusion process $\{S_t, t \geq 0\}$ solving a stochastic differential equation (SDE):

$$dS_t = [r(t) - q(t) + h(S_t, t)]S_t dt + \sigma(S_t, t)S_t dB_t, \quad S_0 = S > 0, \tag{2.1}$$

where $r(t) \geq 0$, $q(t) \geq 0$ are the time-dependent risk-free interest rate and time-dependent dividend yield (assumed to be deterministic functions of time), respectively. The time- and state-dependent functions $\sigma(S_t, t) > 0$ and $h(S_t, t) \geq 0$ are the volatility and default intensity, respectively. For simplicity, we assume that $r(t)$ and $q(t)$ are continuously differentiable in time on $[0, \infty)$ and that $\sigma(S_t, t)$ and $h(S_t, t)$ are continuously differentiable in the stock price and time on $(0, \infty) \times [0, \infty)$. In addition, we also assume that $\sigma(S_t, t)$ and $h(S_t, t)$ remain uniformly bounded as $S \to \infty$. Consequently, the process S does not explode to infinity. On the other hand, we do not impose any restriction as $S \to 0$. Therefore, the process may hit zero depending on the behavior of $\sigma(S_t, t)$ and $h(S_t, t)$ as $S \to 0$. If the process can hit zero in finite time, we kill the process at the first hitting time of zero, $T_0 = \inf\{t \geq 0 : S_t = 0\}$, and is sent to a *cemetery state* Δ, where it remains forever. If zero is unattainable, S has infinite lifetime, and $T_0 = \infty$ by convention. We extend the state space for the process S to include the absorbing state Δ, by $E^{\Delta} = (0, \infty) \cup \{\Delta\}$. We denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the filtration generated by the process S. The term with $h(S_t, t)$ in the drift in (2.1) is needed to compensate for the jump to default to ensure the correct martingale dynamics of the gains process.

The jump to default event is modeled as the first jump time $\tilde{\zeta}$ of a doubly stochastic Poisson process (Cox process) with a jump-to-default *hazard process* $\{\Lambda_t, t \geq 0\}$. That is, $\tilde{\zeta}$ is the first time when the process Λ is greater or equal to a random level $e \sim Exp(1)$:

$$\tilde{\zeta} = \inf\{t : \Lambda_t \geq e\} \tag{2.2}$$

If $h(S,t)$ remains bounded as $S \to 0$, we define $\Lambda_t = \int_0^t h(S_u, u)du$. If the default intensity $h(S,t)$ explodes at T_0 (i.e., $h(S,t) \to \infty$ as $S \to 0$), we define Λ_t by:

$$\Lambda_t = \begin{cases} \int_0^t h(S_u, u)du, & t < T_0 \\ \infty, & t \geq T_0 \end{cases} . \tag{2.3}$$

At time $\tilde{\zeta}$, the stock jumps to the cemetery (bankruptcy) state Δ, where it remains forever. The cemetery (default) state Δ can be identified with zero by setting $\Delta = 0$. We assume that absolute priority applies and equity holders do not receive any recovery in the event of default. Thus, the stock price subject to bankruptcy is a diffusion process $S^\Delta = \{S_t^\Delta, t \geq 0\}$ with the extended state space E^Δ. We call S_t^Δ the *defaultable stock process.* We notice that, in general, in this model default can happen either at time T_0 via diffusion to zero or at time $\tilde{\zeta}$ via a jump to default, whichever comes first. The time of default ζ (lifetime of the process S^Δ in the terminology of Markov processes) is then decomposed into a predictable and a totally inaccessible part:

$$\zeta = T_0 \wedge \tilde{\zeta} \tag{2.4}$$

In our notation, $\{S_t, t \geq 0\}$ is the pre-default stock price process (2.1), while $\{S_t^\Delta, t \geq 0\}$ is the defaultable stock price process, so that $S_t^\Delta = S_t$ for $t < \zeta$ and $S_t^\Delta = \Delta$ for all $t \geq \zeta$. We also notice that if $h(S,t) \to \infty$ as $S \to 0$, it is possible that the diffusion process S^Δ with killing at rate $h(S,t)$ never hits zero, while the process S (2.1) without killing can hit zero. Intuitively, as the jump to default intensity increases towards infinity as the process diffuses towards zero, the process will be almost surely killed through a jump to default from a positive value (will be sent to the cemetery state Δ from a positive value) before it has the opportunity to diffuse down to zero. In such case, $\tilde{\zeta} < T_0$, $\zeta = \tilde{\zeta}$, and $S_{\zeta-}^\Delta > 0$ almost surely.

To keep track of how information is revealed over time, we introduce a default indicator process $\{D_t, t \geq 0\}$, $D_t = \mathbf{1}_{\{t \geq \zeta\}}$, a filtration $\mathbb{D} = \{\mathcal{D}_t, t \geq 0\}$ generated by D, and an enlarged filtration $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ (recall that $\mathbb{F}$ is the filtration generated by the pre-default process S).

If we identify the cemetery state $\Delta = 0$, then we can write the process for the stock price subject to bankruptcy in the form

$$dS_t^\Delta = S_t^\Delta((r(t) - q(t))dt + \sigma(S_{t-}^\Delta, t)dB_t - dM_t)$$

where the compensated default indicator process M_t is a martingale,

$$M_t = D_t - \int_0^{t \wedge \zeta} h(S_u, u)du,$$

where $\int_0^{\tau \wedge t} h(S_u, u)du$ is the $\mathbb{G}$-compensator of D_t. The default hazard rate needs to be added in the drift rate in the pre-default dynamics (2.1) to compensate for the default jump and to ensure that the total expected rate of return to the stockholder is equal to the risk-free rate in the risk-neutral economy, and the discounted gains process is a martingale under the EMM $\mathbb{Q}$. The addition of the jump to default intensity to the drift rate was already discussed in Merton (1976) and more recently by Davis and Lischka (2002).

2.2 Unified Valuation of Corporate Debt, Credit and Equity Derivatives

We view the stock price as the fundamental observable state variable and, within the framework of our reduced-form model, view all securities related to a given firm, such as corporate debt, credit derivatives, and equity derivatives, as contingent claims written on the stock price process (2.1). A fundamental building block for the valuation of contingent claims on defaultable stock is the (risk-neutral) *survival probability*, or the probability of no default prior to time T, conditional on the information available at time $0 \leq t < T$:

$$\mathbb{Q}(\zeta > T|\mathcal{G}_t) = \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\Big[\mathbf{1}_{\{\tilde{\zeta}\wedge T_0>T\}}|\mathcal{G}_t\Big]$$

$$= \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\Big[\mathbf{1}_{\{\tilde{\zeta}>T\}}\mathbf{1}_{\{T_0>T\}}|\mathcal{G}_t\Big] = \mathbf{1}_{\{\zeta>t\}}\mathbb{E}\Big[e^{-\int_t^T h(S_u,u)du}\mathbf{1}_{\{T_0>T\}}|\mathcal{F}_t\Big]. \quad (2.5)$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ is the enlarged filtration of the previous section. The last equality of (2.5) holds since $\mathbf{1}_{\{T_0>T\}}$ is $\mathcal{F}_T$-measurable (see Bielecki and Rutkowski (2004, Corollary 5.1.1) and Jeanblanc et al. (2009, Corollary 7.3.4.2))

The price of a recovery claim that pays a fixed recovery amount R at maturity T if default occurs by T is expressed in terms of the survival probability $\mathbb{Q}(\zeta > T|\mathcal{G}_t)$:

$$Re^{-\int_t^T r(u)du}[1 - \mathbb{Q}(\zeta > T|\mathcal{G}_t)]. \quad (2.6)$$

Then the price of a defaultable zero-coupon bond with unit face value and with recovery amount R $(0 \leq R \leq 1)$ paid at maturity T if default occurs by time T is given by:

$$Re^{-\int_t^T r(u)du}[1 - \mathbb{Q}(\zeta > T|\mathcal{G}_t)] + e^{-\int_t^T r(u)du}\mathbb{Q}(\zeta > T|\mathcal{G}_t). \quad (2.7)$$

A European contingent claim with maturity at time $T > 0$ and payoff $\Psi(S_T)$ at T, given no default by T, and no recovery if default occurs by T, is valued according to:

$$\mathbf{1}_{\{\zeta>t\}}e^{-\int_t^T r(u)du}\mathbb{E}\Big[\Psi(S_T)\mathbf{1}_{\{\zeta>T\}}|\mathcal{G}_t\Big]$$

$$= \mathbf{1}_{\{\zeta>t\}}e^{-\int_t^T r(u)du}\mathbb{E}\Big[e^{-\int_0^T h(S_u,u)du}\Psi(S_T)\mathbf{1}_{\{T_0>T\}}|\mathcal{F}_t\Big]. \quad (2.8)$$

A recovery claim that pays a fixed recovery at the default time ζ, as opposed to at maturity, is valued according to:

$$\mathbf{1}_{\{\zeta>t\}}\mathbb{E}\Big[e^{-\int_t^\zeta r(u)du}\mathbf{1}_{\{\zeta\leq T\}}|\mathcal{G}_t\Big]$$

$$= \mathbf{1}_{\{\zeta>t\}}\int_t^T e^{-\int_t^u r(v)dv}\mathbb{E}\Big[e^{-\int_t^u h(S_v,v)dv}h(S_u,u)\mathbf{1}_{\{T_0>u\}}|\mathcal{F}_t\Big]$$

$$+\mathbf{1}_{\{\zeta>t\}}\mathbb{E}\Big[e^{-\int_t^{T_0}\big(r(u)+h(S_u,u)\big)du}\mathbf{1}_{\{t<T_0\leq T\}}|\mathcal{F}_t\Big]. \quad (2.9)$$

This contingent claim models the CDS protection payoff.

2.3 Equity Options

A *European call option* with strike $K > 0$ with the payoff $\Psi(S_T) = (S_T - K)^+$ at expiration T has no recovery if the firm defaults. A *European put option* with strike $K > 0$ with the payoff $\Psi(S_T) = (K - S_T)^+$ can be decomposed into two parts: the put payoff $(K - S_T)^+ \mathbf{1}_{\{\zeta > T\}}$, given no default by time T, and a recovery payment equal to the strike K at expiration in the event of default $\zeta \leq T$. Assuming no default by time $t \in [0, T)$, the pricing formulas for European-style call and put options take the form:

$$C(S_t, t; K, T) = e^{-\int_t^T r(u)du} \mathbb{E}\Big[e^{-\int_t^T h(S_u,u)du} (S_T - K)^+ \mathbf{1}_{\{T_0 > T\}} | \mathcal{F}_t\Big], \tag{2.10}$$

and

$$P(S_t, t; K, T) = P_0(S_t, t; K, T) + P_D(S_t, t; K, T), \tag{2.11}$$

where

$$P_0(S_t, t; K, T) = e^{-\int_t^T r(u)du} \mathbb{E}\Big[e^{-\int_t^T h(S_u,u)du} (K - S_T)^+ \mathbf{1}_{\{T_0 > T\}} | \mathcal{F}_t\Big], \tag{2.12}$$

and

$$P_D(S_t, t; K, T) = K e^{-\int_t^T r(u)du} [1 - \mathbb{Q}(\zeta > T | \mathcal{G}_t)], \tag{2.13}$$

respectively. One notes that the put pricing formula (2.11) consists of two parts: the present value $P_0(S_T, t; K, T)$ of the put payoff conditional on no default given by Eq. (2.12) (this can be interpreted as the down-and-out put with the down-and-out barrier at zero), as well as the present value $P_D(S_T, t; K, T)$ of the cash payment equal to the strike K in the event of default given by Eq. (2.13). The recovery part of the put is a European-style default claim, a credit derivative that pays a fixed cash amount equal to the strike K at maturity T if and only if the underlying firm has defaulted by time $t < \zeta \leq T$. *Thus, the put option contains an embedded credit derivative.*

2.4 Equity Default Swaps

Equity default swaps (EDS) are a class of hybrid credit-equity products that link credit default swaps (CDS) to equity derivatives with barriers. An EDS is structured similarly to the CDS, with the transparency of the former based on the observable stock price. That is, an EDS delivers a protection payment to the EDS buyer at the time of the triggering event ζ_L, defined as the stock price decline below a pre-specified lower triggering barrier level L. In exchange, the EDS buyer makes periodic premium payments at time intervals δ at the equity default swap rate ϱ up to the triggering event time ζ_L or the final maturity T, whichever comes first. If the triggering event occurs mid-period between the two premium payments, the buyer pays the accrued interest from the time of the last premium payment up to the time of the triggering event. The protection payment is the specified percentage $(1 - R)$ of the EDS notional amount $\mathcal{N}$ (by analogy with the CDS, here R is the "recovery rate" and $1 - R$ is the "loss-given-default", or rather the "loss-given-the-triggering barrier crossing event", that the EDS seller pays to the EDS buyer). The valuation problem is to determine the swap rate ϱ so that the present value of the EDS contract is zero at the contract inception. This swap rate equates the present value of the protection payoff to the present value of the premium payments including accrued interest up to the triggering event:

$$PV(Protection) = PV(Premium + \text{ } Acc.\ Int.).$$

Mathematically, the problem of finding the arbitrage-free swap rate ϱ is reduced to investigating the first passage time ζ_L of a JDED process through a barrier level L. Since the stock price process evolves continuously except for possibly a single jump to zero (jump to default), the first passage time ζ_L of the stock price through a lower barrier $L > 0$ is the first time in which the pre-default diffusion hits the barrier level L or the time of the jump to default $\tilde{\zeta}$, whichever occurs first. That is, the triggering event time is given by $\zeta_L = T_L \wedge \tilde{\zeta}$. Evidently, $\zeta_{L_1} \le \zeta_{L_2}$ for $L_1 > L_2$ and, in particular, $\zeta_L \le \zeta_0$ for $L > 0$. Thus, the CDS rate is the lower bound for EDS rates with positive triggering barriers.

Let N be the total number of premium payments, $\delta = T/N$ be the time between premium payments, and $t_i = i\delta$ with $i = 1, ..., N$ be ith periodic premium payment date. Then, the time zero value of the protection payoff is given by:

$$PV(Protection) = (1-R)\cdot \mathbb{E}\left[e^{-\int_0^{\zeta_L} r(u)du}\mathbf{1}_{\{\zeta_L \le T\}}\right]$$

$$= (1-R)\cdot \Big(\int_0^T e^{-\int_0^u r(v)dv}\mathbb{E}_{0,S_0}\left[e^{-\int_0^{T_L}\left(r(u)+h(S_u,u)\right)du}\mathbf{1}_{\{T_L\le T\}}\right]$$

$$+\mathbb{E}_{0,S_0}\left[e^{-\int_0^u h(S_v,v)dv}h(S_u,u)\mathbf{1}_{\{T_L>u\}}\right]du\Big).$$

The first term in parenthesis is the present value of the payoff triggered by a jump-to-default from a positive stock price, if it occurs prior to maturity and prior to hitting zero via diffusion. The second term is the present value of the payoff if the stock price hits the barrier level L via diffusion, if it occurs prior to maturity and prior to the jump-to-default.

The present value of periodic premium payments up to ζ_L is given by:

$$PV(Premium) = \varrho\cdot\delta\sum_{i=1}^{N} e^{-\int_0^{t_i} r(u)du}\mathbb{E}_{0,S_0}\left[e^{-\int_0^{t_i} h(S_u,u)du}\mathbf{1}_{\{T_L\ge t_i\}}\right].$$

If the triggering event occurs between the two periodic payments dates t_i and t_{i+1} (i.e., $\zeta_L \in (t_i, t_{i+1})$), the EDS buyer pays the interest accrued since the previous payment date t_i up to the triggering event time ζ_L. The expression for the present value of the accrued interest is given in Mendoza-Arriaga and Linetsky (2009) and we do not reproduce it here.

To explicitly evaluate the protection and premium legs of the EDS contract requires one to solve the first passage time problem for the pre-default diffusion process, as the first hitting time T_L enters these expressions. Mendoza-Arriaga and Linetsky (2009) obtain analytical solutions under the Jump to Default Extended CEV process with time-homogeneous parameters by applying the eigenfunction expansion approach.

3 The Jump-to-Default Extended CEV Model (JDCEV)

To be consistent with the leverage effect and the implied volatility skew, in the CEV model the instantaneous volatility is specified as that of a constant elasticity of variance (CEV) process (see Cox (1975), Schroder (1989), Delbaen and Shirakawa (2002), Davydov and Linetsky (2001, 2003), Linetsky (2004c), Jeanblanc et al. (2009, Chapter 6) and Mendoza and Linetsky (2010) for background on the CEV process):

$$\sigma(S,t) = a(t)S^{\beta}, \tag{3.1}$$

where $\beta < 0$ is the volatility elasticity parameter and $a(t) > 0$ is the time-dependent volatility scale parameter. To be consistent with the empirical evidence linking corporate bond yields and CDS spreads to equity volatility, the default intensity in the JDCEV model is specified as an affine function of the instantaneous variance of the underlying stock price:

$$h(S,t) = b(t) + c\,\sigma^2(S,t) = b(t) + c\,a^2(t)S^{2\beta}, \tag{3.2}$$

where $b(t) \geq 0$ is a deterministic non-negative function of time and $c > 0$ is a positive constant parameter governing the sensitivity of h to σ^2. The functions $b(t)$ and $a(t)$ can be determined by reference to given term structures of credit spreads and at-the-money implied volatilities.

For $c \geq 1/2$, the SDE (2.1) with σ and h specified by (3.1)–(3.2) has a unique non-exploding solution. This solution is a diffusion process on $(0,\infty)$ where zero and infinity are both unattainable boundaries. For $c \in (0,1/2)$, infinity is an unattainable (natural) boundary for all $\beta < 0$, while zero is an exit boundary for $\beta \in [c-1/2, 0)$. For $c \in (0,1/2)$ and $\beta < c - 1/2$, zero is a regular boundary, and we specify it as a killing boundary by adjoining a killing boundary condition. Thus, for $c \in (0,1/2)$ the process S can hit zero and is sent to the cemetery state Δ at the first hitting time of zero, T_0. Note that for $c \geq 1/2$, $T_0 = \infty$ since zero is an unattainable boundary. Even though for $c \in (0,1/2)$ the pre-default process S can hit zero, since the intensity $h(S,t)$ goes to infinity as S goes to zero, zero is an unattainable boundary for the process S^Δ killed at the rate $h(S,t)$ ($\tilde{\zeta} < T_0$ a.s. and $\zeta = \tilde{\zeta}$ a.s.). Intuitively, on all sample paths in which the pre-default stock price S diffuses down toward zero, the inverse relationship between h and S causes the process to be killed from a positive value before it can reach zero via diffusion. Consequently, the second term in the expression for the price of the recovery at default Eq. (2.9) vanishes identically in the JDCEV model.

To summarize, in the JDCEV model the defaultable stock is a (time-inhomogeneous) diffusion process $\{S_t^\Delta, t \geq 0\}$ with state space $E^\Delta = (0,\infty) \cup \{\Delta\}$, initial value $S_0 = x > 0$, diffusion coefficient $a(t)x^{\beta+1}$, drift $[r(t)-q(t)+b(t)+c\,a^2(t)x^{2\beta}]x$, and killing rate $h(x,t) = b(t)+c\,a^2(t)x^{2\beta}$. Pre-default, the diffusion solves the JDCEV SDE:

$$dS_t = [r(t) - q(t) + b(t) + c\,a^2(t)X_t^{2\beta}]S_t dt + a(t)S_t^{\beta+1}dB_t, \quad S_0 = x. \tag{3.3}$$

The defaultable stock price process is then $S_t^\Delta = X_t$ for all $t < \zeta$ and $S_t = \Delta$ for all $t \geq \zeta$.

Carr and Linetsky (2006) show that the JDCEV model is fully analytically tractable due to its close connection with the Bessel process (see Revuz and Yor (1999, Chapter XI), Borodin and Salminen (2002), Göing-Jaeschke and Yor (2003) and Jeanblanc et al. (2009, Chapter 6) for background on Bessel processes).

Theorem 3.1 *Let $\{S_t, t \geq 0\}$ be the solution of the JDCEV SDE* (3.3)*. Let $\{R_t^{(\nu)}, t \geq 0\}$ be a Bessel process of index ν and started at x, $BES^{(\nu)}(x)$. Then, for $c \geq 1/2$, the process* (3.3) *can be represented as a re-scaled and time-changed power of a Bessel process:*

$$\{S_t = e^{\int_0^t \alpha(u)du}(|\beta| R_{\tau(t)}^{(\nu)})^{\frac{1}{|\beta|}}, t \geq 0\}, \tag{3.4}$$

where the deterministic time change is:

$$\tau(t) = \int_0^t a^2(u)e^{-2|\beta|\int_0^u \alpha(s)ds}du, \tag{3.5}$$

and

$$\nu = \frac{c - 1/2}{|\beta|} \in \mathbb{R}, \;\; R_0^{(\nu)} = x = \frac{1}{|\beta|} S^{|\beta|} > 0, \;\; \alpha(t) = r(t) - q(t) + b(t). \tag{3.6}$$

For $c \in (0, 1/2)$ ($\nu < 0$), the same representation holds before the first hitting time of zero:

$$\{S_t = e^{\int_0^t \alpha(u)du} (|\beta| R^{(\nu)}_{\tau(t)})^{\frac{1}{|\beta|}}, 0 \le t < T_0^S = \tau^{-1}(T_0^R)\}, \tag{3.7}$$

where $T_0^S = \tau^{-1}(T_0^R)$ ($T_0^R = \tau(T_0^S)$) is the first hitting time of zero for the process S_t ($R_t^{(\nu)}$) and τ^{-1} is the inverse function of the deterministic time change function τ (both processes are killed and sent to the cemetery state Δ at the first hitting time of zero).

Using this reduction to Bessel processes, Carr and Linetsky (2006) show that the survival probability and European call and put options prices in the JDCEV model are expressed in closed form in terms of the non-central chi-square distribution.

Lemma 3.1 *Let X be a $\chi^2(\delta, \alpha)$ random variable, $\nu = \delta/2 - 1$, $p > -(\nu + 1)$, and $k > 0$. The $p-$th moment and truncated $p-$th moments are given by:*

$$\mathcal{M}(p; \delta, \alpha) = E^{\chi^2(\delta,\alpha)}[X^p] = 2^p e^{-\frac{\alpha}{2}} \frac{\Gamma(p + \nu + 1)}{\Gamma(\nu + 1)} \, {}_1F_1(p + \nu + 1, \nu + 1, \alpha/2), \tag{3.8}$$

$$\Phi^+(p, k; \delta, \alpha) = E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X>k\}}] = 2^p \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}} \left(\frac{\alpha}{2}\right)^n \frac{\Gamma(\nu + p + n + 1, k/2)}{n!\Gamma(\nu + n + 1)}, \tag{3.9}$$

$$\Phi^-(p, k; \delta, \alpha) = E^{\chi^2(\delta,\alpha)}[X^p \mathbf{1}_{\{X\le k\}}] = 2^p \sum_{n=0}^{\infty} e^{-\frac{\alpha}{2}} \left(\frac{\alpha}{2}\right)^n \frac{\gamma(\nu + p + n + 1, k/2)}{n!\Gamma(\nu + n + 1)}, \tag{3.10}$$

where $\Gamma(a)$ is the standard Gamma function, $\gamma(a, x) = \int_0^x y^{a-1} e^{-y} dy$ is the incomplete Gamma function, $\Gamma(a, x) = \Gamma(x) - \gamma(a, x)$ is the complementary incomplete Gamma function, and:

$${}_1F_1(a, b, x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \;\; \text{where } (a)_0 = 1, \;\; (a)_n = a(a + 1)...(a + n - 1), \;\; n > 0,$$

is the Kummer confluent hypergeometric function.

The JDCEV survival probability and call and put option pricing formulas are then expressed in terms of these moments and truncated moments of the non-central chi-square distribution.

Theorem 3.2 *Let x and ν be as defined in (3.6), and define $\nu_+ = \nu + 1/|\beta|$, $\delta_+ = 2(\nu_+ + 1)$ and $\tau = \tau(t, T) = \int_t^T a^2(u) e^{-2|\beta| \int_t^u \alpha(s)ds} du$. Assume that default has not happened by time $t \ge 0$, i.e., $\zeta > t$, and $S_t = S > 0$.*
(i) The risk-neutral survival probability (2.5) is given by:

$$\mathbb{Q}(\zeta > T | \mathcal{G}_t) = Q(S, t; T) = e^{-\int_t^T b(u)du} \left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau}\right). \tag{3.11}$$

(ii) The claim that pays one dollar at the time of jump to default ζ (first part of (2.9)) is given by:

$$\int_t^T e^{-\int_t^u [r(s)+b(s)]ds} \left\{ b(u) \left(\frac{x^2}{\tau(t, u)}\right)^{\frac{1}{2|\beta|}} \mathcal{M}\left(-\frac{1}{2|\beta|}; \delta_+, \frac{x^2}{\tau(t, u)}\right) \right.$$

$$+ca^2(u)S^{2\beta}e^{-2|\beta|\int_t^u \alpha(s)ds}\left(\frac{x^2}{\tau(t,u)}\right)^{\frac{1}{2|\beta|}+1}\mathcal{M}\left(-\frac{1}{2|\beta|}-1;\delta_+,\frac{x^2}{\tau(t,u)}\right)\Bigg\}du \tag{3.12}$$

(the second part of (2.9) *vanishes identically since the JDCEV process is killed by a jump to default before it has the opportunity to diffuse to zero).*
(iii) The call option price (2.10) *is given by:*

$$C(S,t;K,T) = e^{-\int_t^T q(u)du}S\Phi^+\left(0,\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right)$$

$$-e^{-\int_t^T [r(u)+b(u)]du}K\left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}}\Phi^+\left(-\frac{1}{2|\beta|},\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right), \tag{3.13}$$

where

$$k = k(t,T) = \frac{1}{|\beta|}K^{|\beta|}e^{-|\beta|\int_t^T \alpha(u)du}. \tag{3.14}$$

(iv) The price of the put payoff conditional on no default by time T (2.12) *is given by:*

$$P_0(S,t;K,T) = e^{-\int_t^T [r(u)+b(u)]du}K\left(\frac{x^2}{\tau}\right)^{\frac{1}{2|\beta|}}\Phi^-\left(-\frac{1}{2|\beta|},\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right)$$

$$-e^{-\int_t^T q(u)du}S\Phi^-\left(0,\frac{k^2}{\tau};\delta_+,\frac{x^2}{\tau}\right), \tag{3.15}$$

and the recovery part of the put option $P_D(S,t;K,T)$ *is given by Eq.* (2.13) *with the survival probability* (3.11).

Mendoza-Arriaga and Linetsky (2009) obtain analytical solutions for the first passage time problem and the EDS valuation problem for the JDCEV problem in the time-homogeneous case (i.e., assuming the parameters r, q, a and b are time-independent). Vidal-Nunes (2009) obtains analytical approximations for American options in the JDCEV model.

Remark 3.1. In the JDCEV model $\beta < 0$. If one sets $\beta = 0$ and specifies the default intensity by $h(S) = \alpha S^{-p}$, one obtains the jump-to-default extended Black-Scholes. This model has been solved analytically by Linetsky (2006) by different mathematical means than the solution to the JDCEV model of Carr and Linetsky (2006) surveyed here.

4 Introducing Jumps and Stochastic Volatility via Time Changes

4.1 Time Changing Jump-to-Default Extended Diffusions

In this section we survey credit-equity models with state-dependent jumps, local-stochastic volatility and default intensity based on the time changed jump-to-default extended diffusion framework developed by Mendoza-Arriaga et al. (2009). We start by defining the defaultable stock price dynamics under the EMM by:

$$S_t = \mathbf{1}_{\{t<\tau_d\}}e^{\rho t}X_{T_t} \equiv \begin{cases} e^{\rho t}X_{T_t}, & t<\tau_d \\ 0, & t\geq\tau_d \end{cases}. \tag{4.1}$$

where $\{X_t, t \geq 0\}$ is a time-homogeneous diffusion starting from a positive value $X_0 = x > 0$ and solving the SDE:

$$dX_t = [\mu + h(X_t)]X_t\,dt + \sigma(X_t)X_t\,dB_t, \tag{4.2}$$

where $\sigma(x)$ is the state-dependent instantaneous volatility, $\mu \in \mathbb{R}$ is a constant, and $h(x)$ is a function to be used as the killing rate in Eq. (4.4). In this section we restrict our attention to the time-homogeneous case. The functions $\sigma(x)$ and $h(x)$ are assumed Lipschitz continuous on $[\epsilon, \infty)$ for each $\epsilon > 0$, $\sigma(x) > 0$ on $(0, \infty)$, $h(x) \geq 0$ on $(0, \infty)$, and $\sigma(x)$ and $h(x)$ remain bounded as $x \to \infty$. We do not assume that $\sigma(x)$ and $h(x)$ remain bounded as $x \to 0$. Under these assumptions the process X does not explode to infinity, but, in general, may reach zero, depending on the behavior of $\sigma(x)$ and $h(x)$ as $x \to 0$. The SDE (4.2) has a unique solution up to the first hitting time of zero, $T_0 = \inf\{t \geq 0 : X_t = 0\}$. If the process can reach zero, we kill it at T_0 and send it to the cemetery state Δ, where it remains for all $t \geq T_0$ (zero is a *killing boundary*). If zero is an inaccessible boundary, we set $T_0 = \infty$ by convention. We call X the *background diffusion process*.

The process $\{T_t, t \geq 0\}$ is a random *time change* assumed independent of X. It is a right-continuous with left limits increasing process starting at zero, $T_0 = 0$, and with finite expectation, $\mathbb{E}[T_t] < \infty$ for every $t > 0$. We consider two important classes of time changes: *Lévy subordinators* (Lévy processes with positive jumps and non-negative drift) that are employed to introduce jumps, and *absolutely continuous time changes* $T_t = \int_0^t V_u\,du$ with a positive process $\{V_t, t \geq 0\}$ called the *activity rate* that are employed to introduce stochastic volatility. We also consider *composite time changes*

$$T_t = T^1_{T^2_t} \tag{4.3}$$

by time changing a Lévy subordinator T^1_t with an absolutely continuous time change T^2_t.

The stopping time τ_d models the time of default. Let T_0 be the first time the diffusion X reaches zero. Let e be an exponential random variable with unit mean, $e \sim Exp(1)$, and independent of X and T. Define

$$\zeta := \inf\{t \in [0, T_0] : \int_0^t h(X_u)du \geq e\}. \tag{4.4}$$

It can be interpreted as the first jump time of a doubly-stochastic Poisson process with the state-dependent intensity $h(X_t)$ if it jumps before time T_0, or T_0 if there is no jump in $[0, T_0]$. At time ζ we kill the process X and send it to the cemetery state Δ, where it remains for all $t \geq \zeta$. The process X is thus a Markov process with killing with *lifetime* ζ.

After applying the time change T to the process X with lifetime ζ, the lifetime of the time changed process X_{T_t} is:

$$\tau_d := \inf\{t \geq 0 : T_t \geq \zeta\}. \tag{4.5}$$

While the process X_t is in the cemetery state for all $t \geq \zeta$, the time changed process X_{T_t} is in the cemetery state for all times t such that $T_t \geq \zeta$ or, equivalently, $t \geq \tau_d$ with τ_d defined by Eq. (4.5). That is, τ_d defined by Eq. (4.5) is the first time the time changed process X_{T_t} is in the cemetery state. We take τ_d to be the time of default. Since we assume that the stock becomes worthless in default, we set $S_t = 0$ for all $t \geq \tau_d$, so that $S_t = 1_{\{t<\tau_d\}} e^{\rho t} X_{T_t}$.

The *scaling factor* $e^{\rho t}$ with some constant $\rho \in \mathbb{R}$ is introduced in order to ensure that the discounted stock price process is a martingale under Q:

$$\mathbb{E}[S_t] < \infty \text{ for every } t \tag{4.6}$$

and

$$\mathbb{E}[S_{t_2}|\mathcal{F}_{t_1}] = e^{(r-q)(t_2-t_1)} S_{t_1} \text{ for every } t_1 < t_2, \tag{4.7}$$

where $r \geq 0$ is the risk-free interest rate and $q \geq 0$ is the dividend yield. The martingale condition (4.6) - (4.7) imposes some restrictions on the model parameters μ and ρ. Namely, if the time change T is a Lévy subordinator with Laplace exponent ϕ, then μ can be any value such that $\mathbb{E}[e^{\mu T_t}] < \infty$ and $\rho = r - q + \phi(-\mu)$. If T is an absolutely continuous time change or a composite time change, than $\mu = 0$ and $\rho = r - q$.

4.2 Time Changes

A *Lévy Subordinator* $\{T_t, t \geq 0\}$ is a non-decreasing Lévy process with positive jumps and non-negative drift with the Laplace transform

$$\mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)} \tag{4.8}$$

with the Laplace exponent given by the Lévy-Khintchine formula

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\nu(ds) \tag{4.9}$$

with the Lévy measure $\nu(ds)$ satisfying $\int_{(0,\infty)} (s \wedge 1)\nu(ds) < \infty$, with non-negative drift $\gamma \geq 0$, and the transition probability $\mathbb{Q}(T_t \in ds) = \pi_t(ds)$, $\int_{[0,\infty)} e^{-\lambda s}\pi_t(ds) = e^{-t\phi(\lambda)}$. The standard references on subordinators include Bertoin (1996, 1999) and Sato (1999) (see also Geman et al. (2001) for finance applications). A subordinator starts at zero ($T_t = 0$ for $t = 0$), drifts at the constant non-negative drift rate γ, and experiences positive jumps controlled by the Lévy measure $\nu(ds)$ (we exclude the trivial case of constant time changes with $\nu = 0$ and $\gamma > 0$). The Lévy measure ν describes the arrival rates of jumps so that jumps of sizes in some Borel set A bounded away from zero occur according to a Poisson process with intensity $\nu(A) = \int_A \nu(ds)$. If $\int_{\mathbb{R}+} \nu(ds) < \infty$, the subordinator is of compound Poisson type with the Poisson arrival rate $\alpha = \int_{\mathbb{R}+} \nu(ds)$ and the jump size probability distribution $\alpha^{-1}\nu$. If the integral $\int_{\mathbb{R}+} \nu(ds)$ is infinite, the subordinator is of infinite activity. Subordinators are processes of finite variation and, hence, the truncation of small jumps is not necessary in the Lévy-Khintchine formula (4.8).

An absolutely continuous time change $\{T_t, t \geq 0\}$ is defined as the time integral of an activity rate process. A key example is given by the CIR activity rate:

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t,$$

where the standard Brownian motion W is independent of the Brownian motion B driving the SDE (4.2), the activity rate process starts at some positive value $V_0 = v > 0$, $\kappa > 0$ is the rate of mean reversion, $\theta > 0$ is the long-run activity rate level, $\sigma_V > 0$ is the activity rate volatility, and it is assumed that the Feller condition is satisfied $2\kappa\theta \geq \sigma_V^2$ to ensure that the process never hits zero. Due to the Cox, Ingersoll and Ross (1985) result giving the closed form solution for the zero-coupon bond in the CIR interest rate model, we have a closed form expression for the Laplace transform of the time change:

$$\mathcal{L}_v(t, \lambda) = E_v[e^{-\lambda \int_0^t V_u du}] = A(t, \lambda)e^{-B(t,\lambda)v}, \tag{4.10}$$

where $V_0 = v$ is the initial value of the activity rate process, $\varpi = \sqrt{2\sigma_V^2\lambda + \kappa^2}$, and

$$A(t,\lambda) = \left(\frac{2\varpi e^{(\varpi+\kappa)t/2}}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi}\right)^{\frac{2\kappa\theta}{\sigma_V^2}}, \quad B(t,\lambda) = \frac{2\lambda(e^{\varpi t}-1)}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi}.$$

Carr et al. (2003) use the CIR activity rate process to time change Lévy processes to introduce stochastic volatility in the popular Lévy models, such as VG, NIG, CGMY.

For the composite time change, $T_t = T^1_{T^2_t}$, where T^1_t is a subordinator with Laplace exponent ϕ and T^2_t is an absolutely continuous time change, by conditioning on T^2, the Laplace transform of the composite time change is:

$$\mathbb{E}[e^{-\lambda T_t}] = \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\lambda T^1_{T^2_t}\right)\big| T^2_t\right]\right] = \mathbb{E}[e^{-T^2_t\phi(\lambda)}] = \mathcal{L}_z(t,\phi(\lambda)). \tag{4.11}$$

4.3 Pricing Corporate Debt, Credit and Equity Derivatives

We begin with the (risk-neutral) survival probability. Conditioning on the time change, we have:

$$\mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t) = \int_0^\infty \mathbb{Q}(\zeta > s)\pi_t(ds) = \int_0^\infty P_s(x,(0,\infty))\pi_t(ds), \tag{4.12}$$

where $P_t(x,(0,\infty)) = \mathbb{Q}(\zeta > t)$ is the survival probability for the diffusion X with lifetime ζ (transition probability for the Markov process X with lifetime ζ from the state $x > 0$ into $(0,\infty)$, $P_t(x,(0,\infty)) = 1 - P_t(x,\{\Delta\})$) and $\pi_t(ds)$ is the probability distribution of the time change T_t. If the survival probability for X and the distribution of the time change $\pi_t(ds)$ are known in closed form, we can obtain the survival probability by integration. The survival probability (4.12) of the defaultable stock price (4.1) can be interpreted as the survival probability up to a random time T_t of the background process X.

A European-style contingent claim with the payoff $\Psi(S_t)$ at maturity $t > 0$, given no default by time t, and no recovery if default occurs, is valued by conditioning on the time change similar to the calculation for the survival probability:

$$e^{-rt}\mathbb{E}[\mathbf{1}_{\{\tau_d>t\}}\Psi(S_t)] = e^{-rt}\mathbb{E}[\mathbf{1}_{\{\zeta>T_t\}}\Psi(e^{\rho t}X_{T_t})] = e^{-rt}\int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}\Psi(e^{\rho t}X_s)\right]\pi_t(ds). \tag{4.13}$$

In particular, for call and put options we have:

$$C(x;K,t) = e^{-rt}\mathbb{E}[(e^{\rho t}X_{T_t} - K)^+\mathbf{1}_{\{\tau_d>t\}}] = e^{-rt}\int_0^\infty \mathbb{E}[(e^{\rho t}X_s - K)^+\mathbf{1}_{\{\zeta>s\}}]\pi_t(ds), \tag{4.14}$$

and

$$P(x;K,t) = P_0(x;K,t) + P_D(x;K,t), \tag{4.15}$$

where

$$P_0(x;K,t) = e^{-rt}\int_0^\infty \mathbb{E}[(K - e^{\rho t}X_s)^+\mathbf{1}_{\{\zeta>s\}}]\pi_t(ds) \tag{4.16}$$

and

$$P_D(x;K,t) = Ke^{-rt}[1 - \mathbb{Q}(\tau_d > t)], \tag{4.17}$$

respectively.

Calculating expectations in the above expressions involves first computing the expectation $\mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}f(X_s)\right]$ for the background diffusion X and then integrating the result in time with the distribution of the time change T_t. For general Lévy subordinators and for the integral of the CIR process the distributions are not available in closed form. Instead, their Laplace transforms are available (Eqs. (4.8), (4.10) and (4.11)). In principle, the distribution can be recovered by numerically inverting the Laplace transform. The second step is then to compute the integral in Eqs. (4.12) and (4.13). Thus, even if we can determine the expectation $\mathbb{E}\left[\mathbf{1}_{\{\zeta>s\}}f(X_s)\right]$ for the background diffusion X in closed form, we still need to perform double numerical integration in order to compute (4.13) for the time changed process.

Mendoza-Arriaga et al. (2009) propose two alternative approaches that avoid any need for Laplace inversion and numerical integration and are based on the resolvent operator and the spectral expansion, respectively. We now briefly summarize these two approaches that yield explicit analytical representations of the expectation operator

$$\mathcal{P}_t f(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\zeta>t\}}f(X_t)\right] \tag{4.18}$$

associated with a one-dimensional diffusion X with lifetime ζ that are in the form suitable to perform time changes (the time t enters only through the exponentials).

Remark 4.1. The two approaches presented in sections 4.4 and 4.5 rely on properties of *one-dimensional time-homogeneous* diffusion processes. In particular, section 4.5 requires that the semigroup $\{\mathcal{P}_t, t \geq 0\}$ of expectation operators associated with the diffusion is symmetric in the Hilbert space of functions on $(\ell, r) \subseteq \mathbb{R}$ square-integrable with the speed density of the diffusion process X.

4.4 Valuation of Contingent Claims on Time Changed Markov Processes: A Resolvent Operator Approach

We start by defining the *resolvent operator* $\mathcal{R}_s$ (e.g., Ethier and Kurtz (1986)) as the Laplace transform of the expectation operator (4.18),

$$\mathcal{R}_s f(x) := \int_0^\infty e^{-st}\mathcal{P}_t f(x)dt. \tag{4.19}$$

For one-dimensional diffusions, the resolvent operator can be represented as an integral operator (e.g., Borodin and Salminen (2002))

$$\mathcal{R}_s f(x) = \int_\ell^r f(y)G_s(x,y)dy = \frac{\phi_s(x)}{w_s}\int_\ell^x f(y)\psi_s(y)\mathfrak{m}(y)dy + \frac{\psi_s(x)}{w_s}\int_x^r f(y)\phi_s(y)\mathfrak{m}(y)dy, \tag{4.20}$$

where the *resolvent kernel* or *Green's function* $G_s(x,y)$ is the Laplace transform of the transition probability density, $G_s(x,y) = \int_0^\infty e^{-st}p(t;x,y)dt$. It admits the following explicit representation:

$$G_s(x,y) = \frac{\mathfrak{m}(y)}{w_s}\begin{cases} \psi_s(x)\phi_s(y), & x \leq y \\ \psi_s(y)\phi_s(x), & y \leq x \end{cases}, \tag{4.21}$$

where the function $\mathfrak{m}(x)$ is the *speed density* of the diffusion process X and is constructed from the diffusion and drift coefficients, $a(x)$ and $b(x)$ as follows (see Borodin and Salminen (2002), p.17):

$$\mathfrak{m}(x) = \frac{2}{a^2(x)\mathfrak{s}(x)}, \text{ where } \mathfrak{s}(x) = \exp\left(-\int_{x_0}^{x} \frac{2b(y)}{a^2(y)}dy\right), \tag{4.22}$$

where $x_0 \in (\ell, r)$ is an arbitrary point in the state space (ℓ, r) of the 1D diffusion. When X is a pre-default stock price process, the state space is $(0, \infty)$. The function $\mathfrak{s}(x)$ is called the *scale density* of the diffusion process X.

For $s > 0$, the functions $\psi_s(x)$ and $\phi_s(x)$ can be characterized as the unique (up to a multiplicative factor independent of x) solutions of the *Sturm-Liouville equation* associated with the infinitesimal generator $\mathcal{G}$ of the one-dimensional diffusion process,

$$\mathcal{G}u(x) = \frac{1}{2}a^2(x)\frac{d^2u}{dx^2}(x) + b(x)\frac{du}{dx}(x) - c(x)u(x) = su(x), \tag{4.23}$$

by firstly demanding that $\psi_s(x)$ is increasing in x and $\phi_s(x)$ is decreasing, and secondly imposing boundary conditions at regular boundary points. The functions $\psi_s(x)$ and $\phi_s(x)$ are called *fundamental solutions* of the Sturm-Liouville equation (4.23). Their *Wronskian* defined with respect to the scale density is independent of x:

$$w_s = \frac{1}{\mathfrak{s}(x)}(\psi_s'(x)\phi_s(x) - \psi_s(x)\phi_s'(x)). \tag{4.24}$$

In Eq. (4.20) we interchanged the Laplace transform integral in t and the expectation integral in y. This interchange is allowed by Fubini's theorem if and only if the function f is such that

$$\int_{\ell}^{x} |f(y)\psi_s(y)|\mathfrak{m}(y)dy < \infty \text{ and } \int_{x}^{r} |f(y)\phi_s(y)|\mathfrak{m}(y)dy < \infty \;\; \forall \; x \in (\ell, r), \;\; s > 0. \tag{4.25}$$

For f satisfying these integrability conditions, we can then recover the expectation (4.18) by first computing the resolvent operator (4.20) and then inverting the Laplace transform via the Bromwich Laplace transform inversion formula (see Pazy (1983) for the Laplace inversion formula for operator semigroups):

$$\mathcal{P}_t f(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\zeta > t\}} f(X_t)\right] = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st}\mathcal{R}_s f(x)ds. \tag{4.26}$$

A crucial observation is that in the representation (4.26) *time only enters through the exponential* e^{st}. We can thus write:

$$\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t})] = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathbb{E}[e^{sT_t}]\mathcal{R}_s f(x)ds = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathcal{L}(t, -s)\mathcal{R}_s f(x)ds, \tag{4.27}$$

where $\mathcal{L}(t, \lambda) = \mathbb{E}[e^{-\lambda T_t}]$ is the Laplace transform of the time change (here we require that $\mathbb{E}[e^{\varepsilon T_t}] = \mathcal{L}(t, -\varepsilon) < \infty$). We notice that this result does not require the knowledge of the transition probability measure of the time change, and only requires the knowledge of the Laplace transform of the time change (e.g., Eqs. (4.8), (4.10) and (4.11)). Furthermore, it does not require the knowledge of the expectation $\mathbb{E}[\mathbf{1}_{\{\zeta > t\}} f(X_t)]$ for the background process X, and only requires the knowledge of the resolvent $\mathcal{R}_s f(x)$ given by Eq. (4.20). The Laplace transform inversion in (4.27) can be performed by appealing to the Cauchy Residue Theorem to calculate the Bromwich integral in the complex plane.

4.5 Valuation of Contingent Claims on Time Changed Markov Processes: A Spectral Expansion Approach

We observe that the transition probability density $p(t;x,y)$ of a one-dimensional diffusion X can be obtained from the Green's function by inverting the Laplace transform. The Laplace inversion yields the *spectral representation of the transition density* $p(t;x,y)$ originally obtained by McKean (1956) (see also Ito and McKean (1974, Section 4.11), Wong (1964), and Karlin and Taylor (1981), and Linetsky (2004c, 2007) for applications in finance).

Define the inner product $(f,g) := \int_\ell^r f(x)g(x)\mathfrak{m}(x)dx$ and let $L^2((\ell,r),\mathfrak{m})$ be the Hilbert space of functions on (ℓ,r) square-integrable with the speed density $\mathfrak{m}(x)$, i.e., with $\|f\| < \infty$, where $\|f\|^2 = (f,f)$. Then the semigroup $\{\mathcal{P}_t, t \geq 0\}$ of expectation operators associated with a one-dimensional diffusion is symmetric in $L^2((\ell,r),\mathfrak{m})$, i.e., $(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)$ for every $f,g \in L^2((\ell,r),\mathfrak{m})$ and $t \geq 0$. It follows that the infinitesimal generator $\mathcal{G}$ of a symmetric semigroup, as well as the resolvent operators $\mathcal{R}_s$, are self-adjoint, and we can appeal to the Spectral Theorem for self-adjoint operators in Hilbert space to obtain their spectral representations.

In the important special case when the spectrum of $\mathcal{G}$ in $L^2((\ell,r),\mathfrak{m})$ is purely discrete, the spectral representation has the following form. Let $\{\lambda_n\}_{n=1}^\infty$, $0 \leq \lambda_1 < \lambda_2 < ...$, $\lim_{n\uparrow\infty} \lambda_n = \infty$, be the eigenvalues of $-\mathcal{G}$ and let $\{\varphi_n\}_{n=1}^\infty$ be the corresponding eigenfunctions normalized so that $\|\varphi_n\|^2 = 1$. The pair (λ_n, φ_n) solves the *Sturm-Liouville eigenvalue-eigenfunction problem* for the (negative of the) differential operator in (4.23): $-\mathcal{G}\varphi_n = \lambda_n \varphi_n$. Dirichlet boundary condition is imposed at an endpoint if it is a killing boundary. Then the spectral representations for the transition density $p(t;x,y)$ and the expectation operator $\mathcal{P}_t f(x)$ for $f \in L^2((\ell,r),\mathfrak{m})$ take the form of *eigenfunction expansions* (for $t > 0$ the eigenfunction expansion (4.28) converges uniformly on compact squares in $(\ell,r)\times(\ell,r)$):

$$p(t;x,y) = \mathfrak{m}(y)\sum_{n=1}^\infty e^{-\lambda_n t}\varphi_n(x)\varphi_n(y), \tag{4.28}$$

$$\mathcal{P}_t f(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\zeta>t\}} f(X_t)\right] = \sum_{n=1}^\infty c_n e^{-\lambda_n t}\varphi_n(x) \tag{4.29}$$

with the expansion coefficients $c_n = (f,\varphi_n)$ satisfying the Parseval equality $\|f\|^2 = \sum_{n=1}^\infty c_n^2 < \infty$. The eigenfunctions $\{\varphi_n(x)\}_{n=1}^\infty$ form a complete orthonormal basis in the Hilbert space $L^2((\ell,r),\mathfrak{m})$, i.e., $(\varphi_n,\varphi_n) = 1$ and $(\varphi_n,\varphi_m) = 0$ for $n \neq m$. They are also eigenfunctions of the expectation operator, $\mathcal{P}_t\varphi_n(x) = e^{-\lambda_n t}\varphi_n(x)$, with eigenvalues $e^{-\lambda_n t}$, and of the resolvent operator, $\mathcal{R}_s\varphi_n(x) = \varphi_n(x)/(s+\lambda_n)$, with eigenvalues $1/(s+\lambda_n)$.

More generally, the spectrum of the infinitesimal generator $\mathcal{G}$ in $L^2((\ell,r),m)$ may be continuous, in which case the sums in (4.28)-(4.29) are replaced with the integrals. We do not reproduce general results on spectral expansions with continuous spectrum here and instead refer the reader to Davydov and Linetsky (2003), Lewis (1998, 2000), and Linetsky (2004a,b,c, 2007) for further details on the spectral representation for one-dimensional diffusions and their applications in finance.

We notice that in the expression (4.29) *time enters only through the exponentials* $e^{-\lambda_n t}$. Then computing expectations for $f \in L^2((\ell,r),\mathfrak{m})$ can be done as follows:

$$\mathbb{E}[\mathbf{1}_{\{\zeta>T_t\}} f(X_{T_t})] = \sum_{n=1}^\infty c_n \mathbb{E}[e^{-\lambda_n T_t}]\varphi_n(x) = \sum_{n=1}^\infty c_n \mathcal{L}(t,\lambda_n)\varphi_n(x), \tag{4.30}$$

where $\mathcal{L}(t,\lambda)$ is the Laplace transform of the time change. Due to the fact that time enters the spectral expansion only through the exponentials $e^{-\lambda_n s}$, integrating this exponential against the distribution of the time change $\pi_t(ds)$, the integral in s in (4.29) reduces to the Laplace transform of the time change, $\int_{[0,\infty)} e^{-\lambda_n s}\pi_t(ds) = \mathcal{L}(t,\lambda_n)$.

Remark 4.2. A key feature of the resolvent approach and spectral representation approach is that in both methodologies the temporal and spatial variables are separated. Furthermore, since the time variable t enters the expressions (4.26) and (4.29) only through the exponential function ($e^{-\mu t}$, for some μ), then the analytical tractability of $\mathbb{E}[\mathbf{1}_{\{\zeta>T_t\}} f(X_{T_t})]$ is feasible upon knowing the Laplace transform of the time change, $\mathcal{L}(t,\mu)$.

4.6 Time Changing the JDCEV Model

We now assume that the process X in (4.1) follows a time-homogeneous JDCEV process. That is, the parameters r, q, and b are non-negative constants while a is a strictly positive constant. The infinitesimal generator of this diffusion on $(0,\infty)$ has the form:

$$\mathcal{G}f(x) = \frac{1}{2}a^2 x^{2\beta+2}\frac{d^2 f}{dx^2}(x) + (\mu + b + c\,a^2 x^{2\beta})x\frac{df}{dx}(x) - (b + c\,a^2x^{2\beta})f(x). \tag{4.31}$$

When applying a time change T_t to the JDCEV process, the parameter μ is assumed to be such that $\mathbb{E}[e^{\mu T_t}] < \infty$. The parameter ρ in (4.1) is assumed to be such that the stock price process S_t of (4.1) is a discounted martingale. For the JDCEV time changed with a Lévy subordinator with the Laplace exponent $\phi(\lambda)$, this fixes ρ in terms of μ by $\rho = r - q + \phi(-\mu)$. For the JDCEV models with absolutely continuous and composite time changes, this restricts the parameters as follows: $\mu = 0$, $\rho = r - q$.

The scale and speed densities of the JDCEV diffusion are:

$$\mathfrak{m}(x) = \frac{2}{a^2}x^{2c-2-2\beta}e^{Ax^{-2\beta}},\ \ \mathfrak{s}(x) = x^{-2c}e^{-Ax^{-2\beta}},\ \text{ where }\ A := \frac{\mu+b}{a^2|\beta|}. \tag{4.32}$$

The following theorem presents the fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ entering the expression for the Green's function (4.21) and their Wronskian w_s (4.24). Here we present the results for $\mu + b > 0$.

Theorem 4.1 *For a JDCEV diffusion process with the infinitesimal generator* (4.31) *with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$ and such that $\mu + b > 0$, the increasing and decreasing fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ are:*

$$\psi_s(x) = x^{\frac{1}{2}+\beta-c}e^{-\frac{1}{2}Ax^{-2\beta}}M_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{4.33}$$

$$\phi_s(x) = x^{\frac{1}{2}+\beta-c}e^{-\frac{1}{2}Ax^{-2\beta}}W_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{4.34}$$

where $M_{k,m}(z)$ and $W_{k,m}(z)$ are the first and second Whittaker functions with indexes

$$\nu = \frac{1+2c}{2|\beta|},\ \ \varkappa(s) = \frac{\nu-1}{2} - \frac{s+\xi}{\omega},\ \ \text{where } \omega = 2|\beta|(\mu+b),\ \ \xi = 2c(\mu+b)+b, \tag{4.35}$$

and the constant A is defined in Eq. (4.32)*. The Wronskian w_s defined by Eq.* (4.24) *reads:*

$$w_s = \frac{2(\mu+b)\Gamma(1+\nu)}{a^2\Gamma(\nu/2+1/2-\varkappa(s))}. \tag{4.36}$$

The Green's function is given by Eq. (4.21). Inverting the Laplace transform leads to the spectral representation of the transition density (4.28).

Theorem 4.2 *When $\mu+b > 0$, the spectrum of the negative of the infinitesimal generator* (4.31) *is purely discrete with the eigenvalues and eigenfunctions:*

$$\lambda_n = \omega n + \xi, \;\; \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!(\mu+b)}{\Gamma(\nu+n)}} x e^{-Ax^{-2\beta}} L_{n-1}^{(\nu)}(Ax^{-2\beta}), \;\; n = 1, 2, ..., \tag{4.37}$$

where $L_n^{(\nu)}(x)$ are the generalized Laguerre polynomials and ξ and ω are defined in (4.35)*. The spectral representation (eigenfunction expansion) of the JDCEV transition density is given by* (4.28) *with these eigenvalues and eigenfunctions and the speed density* (4.32)*.*

The closed form solutions for the survival probability and call and put options given in Proposition 3.2 are not suitable for time changes since they depend on time in a non-trivial manner. Mendoza-Arriaga et al. (2009) obtain alternative representations based on the theory in sections 4.4 and 4.5 and Theorems 4.1 and 4.2 with time entering only through exponentials.

Theorem 4.3 *For a JDCEV diffusion process with the infinitesimal generator* (4.31) *with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$, $\mu + b > 0$ and started at $x > 0$, the survival probability $\mathbb{Q}(\zeta > t)$ is given by:*

$$\mathbb{Q}(\zeta > t) = \sum_{n=0}^{\infty} e^{-(b+\omega n)t} \frac{\Gamma(1+\frac{c}{|\beta|}) \left(\frac{1}{2|\beta|}\right)_n}{\Gamma(\nu+1) n!} A^{\frac{1}{2|\beta|}} x e^{-Ax^{-2\beta}} \, {}_1F_1\left(1-n+\frac{c}{|\beta|}; \nu+1; Ax^{-2\beta}\right), \tag{4.38}$$

where ${}_1F_1(a;b;x)$ is the confluent hypergeometric function, $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)...(a+n-1)$ is the Pochhammer symbol, and the constants A, ν, and ω are as defined in Theorem 4.1.

The proof is based on first computing the resolvent (4.19) with $f(x) = 1$ and then inverting the Laplace transform (4.26) analytically. Since constants are not square-integrable on $(0, \infty)$ with the speed density (4.32), we cannot use the spectral expansion approach of section 4.5 and instead follow the Laplace transform approach of section 4.4.

We now present the result for the put option. The put option price in the model (4.1) is given by Eqs. (4.15)-(4.17). In particular, in order to compute the price of the put payoff conditional on no default before expiration, $P_0(x; K, t)$, we need to compute the expectation $\mathbb{E}[(K - e^{\rho t} X_s)^+ \mathbf{1}_{\{\zeta > s\}}] = e^{\rho t} \mathbb{E}[(e^{-\rho t} K - X_s)^+ \mathbf{1}_{\{\zeta > s\}}]$ for the JDCEV process (4.31). The put payoff $f(x) = (k - x)^+$ is in the Hilbert space $L^2((0, \infty), \mathfrak{m})$ of functions square-integrable with the speed density (4.32) and, hence, the expectation has a spectral expansion. The eigenfunction expansion coefficients are computed in closed form.

Theorem 4.4 *For a JDCEV diffusion process with the infinitesimal generator* (4.31) *with parameters $\beta < 0$, $a > 0$, $b \geq 0$, $c \geq 0$ and such that $\mu + b > 0$, the expectation $\mathbb{E}[(k - X_t)^+ \mathbf{1}_{\{\zeta > t\}}]$ is given by the eigenfunction expansion* (4.29) *with the eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ given in Theorem 4.2 and expansion coefficients:*

$$c_n = \frac{A^{\nu/2+1} k^{2c+1-2\beta} \sqrt{\Gamma(\nu+n)}}{\Gamma(\nu+1)\sqrt{(\mu+b)(n-1)!}}$$

$$\times \left\{ \frac{|\beta|}{c+|\beta|} \, {}_2F_2 \left(\begin{array}{cc} 1-n, & \frac{c}{|\beta|}+1 \\ \nu+1, & \frac{c}{|\beta|}+2 \end{array} ; Ak^{-2\beta} \right) - \frac{\Gamma(\nu+1)(n-1)!}{\Gamma(\nu+n+1)} L_{n-1}^{(\nu+1)} \left(Ak^{-2\beta} \right) \right\}, \tag{4.39}$$

where ${}_2F_2$ is the generalized hypergeometric function.

The survival probability entering the put pricing formula is already computed in Theorem 4.3. The survival probability for the time changed JDCEV process is immediately obtained from (4.38) by replacing $e^{-(b+\omega n)t}$ with $\mathcal{L}(t, b+\omega n)$ (the Laplace transform of the time change). For the put pricing formula, the factors $e^{-\lambda_n t}$ in the eigenfunction expansion are replaced with $\mathcal{L}(t, \lambda_n)$ (the Laplace transform of the time change evaluated at the eigenvalue; see Eq. (4.30)). The pricing formula for the call option is obtained via the put-call parity.

5 Numerical Illustration

We start with the JDCEV process with $\mu = 0$ and time change it with a composite time change process $T_t = T^1_{T^2_t}$, where T^1 is the Inverse Gaussian (IG) subordinator with the Lévy measure $\nu(ds) = Cs^{-3/2}e^{-\eta s}$ and the Laplace exponent $\phi(s) = \gamma s + 2C\sqrt{\pi}(\sqrt{s+\eta} - \sqrt{\eta})$ and T^2 is the time integral of the activity rate following the CIR process. In order to satisfy the martingale condition, we set $\rho = r - q$. The parameter values in our numerical example are listed in Table 1.

JDCEV

S	a	β	c	b	r	q
50	10	-1	0.5	0.01	0.05	0

CIR

V	θ	σ_V	κ
1	1	1	4

IG

γ	η	C
0	8	$2\sqrt{2/\pi}$

Table 1: *Parameter values.*

The JDCEV process parameter a entering into the local volatility function $\sigma(x) = a\,x^{\beta}$ is selected so that the local volatility is equal to 20% when the stock price is \$50, i.e., $a = 0.2 * 50^{-\beta} = 10$ with $\beta = -1$. We consider a pure jump process with no diffusion component ($\gamma = 0$). For this choice of parameters of the IG time change and the CIR activity rate process the time change has the mean and variance $E[T_1] = 1$ and $Var[T_1] = 1/16$ at $t = 1$.

Figure 1 plots the implied volatility smile/skew curves of options priced under this model for several different maturities. We compute options prices in this model using Theorem 4.4 and then compute implied volatilities of these options by inverting the Black-Scholes formula. We observe that in this model shorter maturity skews are steeper, and flatten out as maturity increases, consistent with empirical observations in options markets. In the time changed JDCEV model short maturity options exhibit a true volatility smile with the increase in implied volatilities both to the right and to the left of the at-the-money strike. This behavior cannot be captured in the pure diffusion model since in the JDCEV the implied volatility skew results from the leverage effect (the local volatility is a decreasing function of stock price) and the possibility of default (the default intensity is a decreasing function of stock price). The resulting implied volatility skew is a decreasing function of strike. After the time change with jumps, the resulting

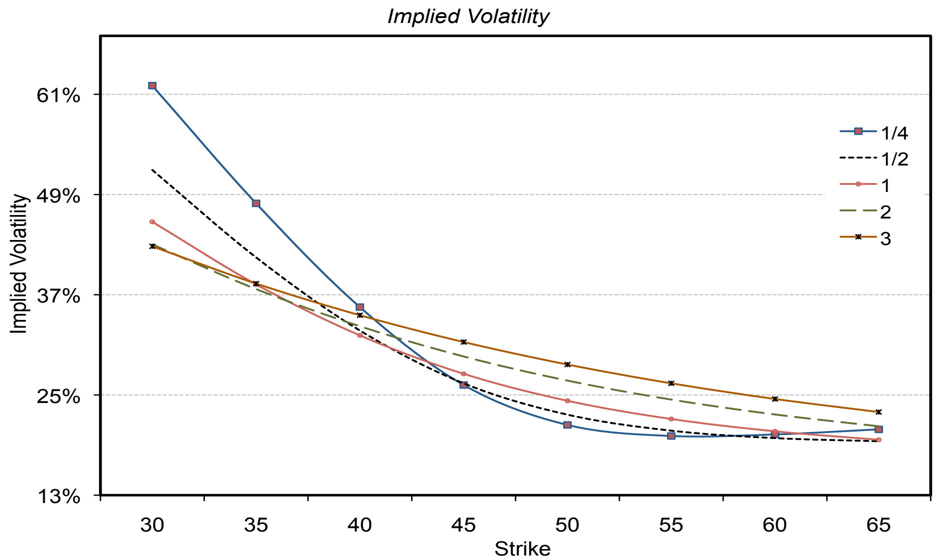

Figure 1: *Implied volatility smile/skew curves as functions of the strike price for times to maturity from 1/4 to 3 years. The current stock price level is 50.*

jump process has both positive and negative jumps. This results in the implied volatility smile pattern with volatility "smiling" on both sides of the at-the-money level.

Figure 2 plots the default probability and the credit spread (assuming zero recovery in default) as functions of time to maturity for several initial levels of the stock price. As the stock price decreases, the credit spreads of all maturities increase, but the shorter and intermediate maturities increase the fastest. This results in a pronounced hump in the term structure of credit spreads around these intermediate maturities. As the stock price falls, the hump becomes more pronounced and shifts towards shorter maturities. This increase in credit spreads with the decrease in the stock price is explained by both the leverage effect through the increase in the local volatility of the original diffusion and, hence, more jump volatility for the jump process after the time change, as well as the increase in the default intensity of the original diffusion process and the jump process after the time change.

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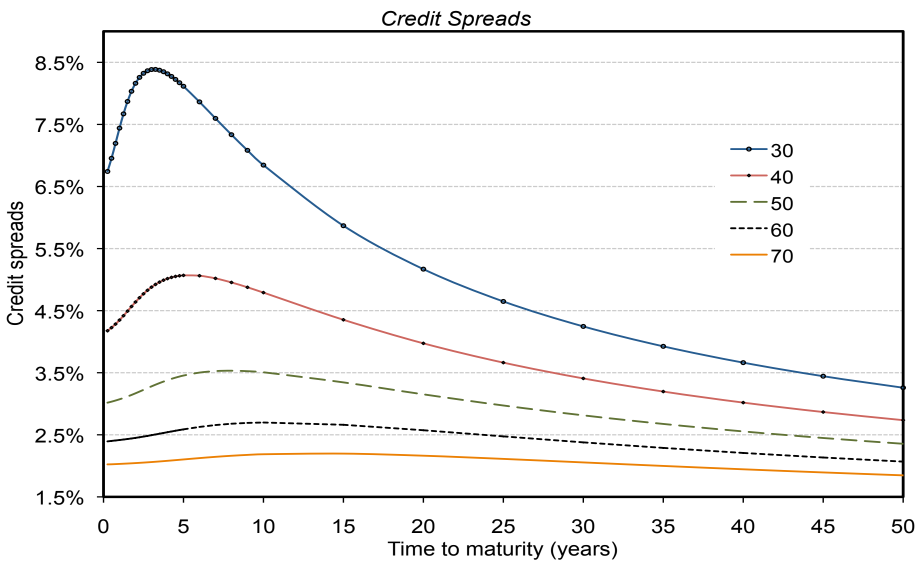

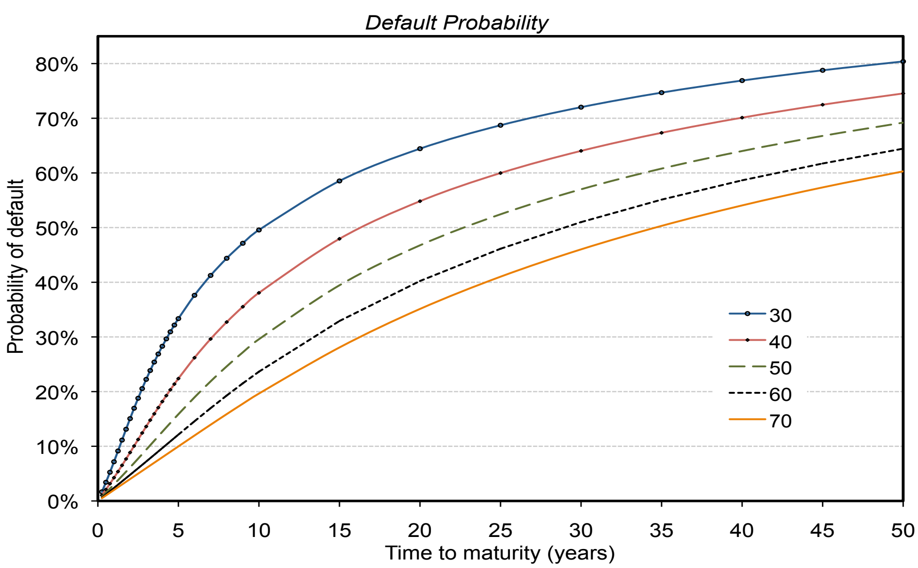

Figure 2: *Credit spreads and default probabilities as functions of time to maturity for the initial stock price levels* $S = 30,\ 40,\ 50,\ 60,\ 70$.

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