Variance swaps on defaultable assets and market implied time-changes

Matthew Lorig * Oriol Lozano-Carbassé † Rafael Mendoza-Arriaga ‡

February 27, 2015

Abstract

We compute the value of a variance swap when the underlying is modeled as a Markov diffusion process time changed by a Lévy subordinator. In this framework, the underlying may exhibit jumps with a state-dependent Lévy measure, local stochastic volatility and have a local stochastic default intensity. Moreover, the Lévy subordinator that drives the underlying can be obtained directly by observing European call/put prices. To illustrate our general framework, we provide an explicit formula for the value of a variance swap when the underlying is modeled as a Lévy subordinated Jump-to-default CEV process (see Carr and Linetsky (2006)). In this example, we extend the results of Mendoza-Arriaga et al. (2010), by allowing for joint valuation of credit and equity derivatives as well as variance swaps.

1 Introduction

A *variance swap* (VS) is a forward contract written on the realized variance of an underlying $S = \{S_t, t \geq 0\}$. As is typical in derivatives literature, we define the *realized variance* over the interval $[0, t]$ as $[\log S]_t$, the continuously sampled quadratic variation of the $\log S$. Thus, at the maturity date t, the long side of a VS has a payoff of

$$[\log S]_t - K_{\text{var}}. \tag{1.1}$$

The *variance swap rate* K_{var} is a constant determined at inception so that the initial value of the VS is zero. It follows from no-arbitrage arguments that $K_{\text{var}} = \mathbb{E}([\log S]_t)$, where $\mathbb{E}$ denotes expectation under the (market's chosen) risk-neutral pricing measure $\mathbb{P}$.

Note that the floating leg $[\log S]_t$ of a VS is only well-defined for a process S that is strictly positive. Since the majority of VS contracts are written on indices that have a negligible probability of going to zero, this is not typically an issue. However, for single name VS contracts, the possibility of default is a non-negligible risk and must be accounted for. The first major contribution of this paper is to propose a modified VS contract, whose floating leg is given by $[\log S]_{t\wedge\zeta^\phi-}$, where ζ^ϕ is the default time of S (i.e., the fist time S hits zero). For any asset that cannot default, the payoff of the modified VS coincides with the traditional VS. For an asset that can default from a strictly positive value $S_{\zeta^\phi-} > 0$, the payoff of the modified VS is well-defined and finite.

The second significant contribution of this paper is to derive an expression for the modified VS rate $K_{\text{var}} := \mathbb{E}([\log S]_{t\wedge\zeta^\phi-})$ assuming the underlying S is modeled by time-changing a killed diffusion with a Lévy subordinator. This flexible class of processes, first introduced in Mendoza-Arriaga et al. (2010), allows S to experience state-dependent jumps, local volatility, and the possibility of default – all of which are important when modeling single names. The third important development in this paper is to provide the canonical decomposition of S (i.e., the Lévy-Itô SDE for S). The canonical decomposition of S is not provided in Mendoza-Arriaga et al. (2010) and is needed for quadratic variation computation.

The fourth and final notable contribution of this paper is to provide a semi-parametric calibration procedure for the Lévy subordinated (killed) diffusion. Specifically, for a parametrically-defined diffusion, we

*Department of Applied Mathematics, University of Washington at Seattle.
†Bendheim Center for Finance, 26 Prospect Avenue, Princeton University, Princeton, NJ 08540, United States.
‡McCombs School of Business, University of Texas at Austin.

show how to obtain its model parameters from liquid European call and put prices with no knowledge of the subordinator. Then, after obtaining the parameters of the diffusion, we demonstrate how to use call and put prices to obtain the Laplace exponent of the subordinator *nonparametrically*. We further show how to obtain the drift, jump intensity and Lévy measure of the subordinator from the Laplace exponent.

The rest of this paper proceeds as follows. In Section 2 we review the class of Lévy subordinated killed diffusion processes and derive the canonical representation for processes in this class. In Section 3 we introduce a modified VS contract, which is well-defined for process that can jump to zero. We also derive in Section 3 a general expression for the value of the modified VS in the Lévy subordinated killed diffusion setting. In Section 4 we briefly review some results concerning eigenfunction expansions, which will be needed in subsequent sections. In Section 5, we compute VS rates under the subordinated jump-to-default constant elasticity of variance (JDCEV) process' specification. In Section 6 we present our semi-parametric calibration procedure and introduce the notion of a market implied time change. In Section 7, we perform a series of numerical experiments, which are designed to test the applicability and robustness of our calibration procedure. We also compare our pricing and calibration methodology to some alternative methods. Finally, in Section 8, we provide some concluding remarks and discuss directions for future research. Long proofs are given in the Appendix.

2 The stock price process

In this section we construct a model for the stock price process S under the *subordinate bivariate framework* introduced in Mendoza-Arriaga et al. (2010) and further developed in Mendoza-Arriaga and Linetsky (2014b). With the exception of Theorem 2.1, which characterizes S as a martingale and provides its canonical representation, the content of this section can be found with more detail in Mendoza-Arriaga and Linetsky (2014b). Here, we simply summarize the results that are needed to establish Theorem 2.1.

We assume frictionless markets, zero interest rates, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as given. All stochastic processes defined below live on this probability space, and all expectations are with respect to $\mathbb{P}$ unless stated otherwise. As all of the processes considered below are of the Feller type, the natural filtrations generated by these processes are right-continuous. We shall assume that all the (natural) filtrations defined throughout this Section are augmented to include the $\mathbb{P}$-null sets. In particular, this assumption holds for the *subordinate filtration* $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, which we describe at the end of this section.

As in Mendoza-Arriaga et al. (2010), we model the stock price dynamics under the risk-neutral pricing measure $\mathbb{P}$ as a stochastic process $S = \{S_t, t \geq 0\}$ defined by

$$S_t = (1 - D_{T_t})e^{\rho t} X_{T_t}, \qquad D_t = \mathbb{I}_{\{\zeta \leq t\}}, \qquad X_0 = x, \qquad T_0 = 0, \qquad \zeta > 0. \tag{2.1}$$

Here, the background process $X = \{X_t, t \geq 0\}$ is a scalar Feller diffusion, $T = \{T_t, t \geq 0\}$ is a Lévy subordinator, ρ is a scaling factor, which is needed to ensure that the asset price S is a martingale, and ζ is a positive random variable, which will be used to model a possible default event of the underlying S. We now describe each of the above-mentioned elements in detail:

Background Feller process X. $X = \{X_t, t \geq 0\}$ is a time-homogeneous Markov diffusion process, starting from a positive value $X_0 = x > 0$, which solves a stochastic differential equation (SDE) of the form

$$dX_t = b(X_t)\, dt + a(X_t)\, dB_t, \quad \text{with} \quad b(x) := [\mu + k(x)]x \quad \text{and} \quad a(x) := \sigma(x)x. \tag{2.2}$$

Here, $\sigma(x)$ and $[\mu + k(x)]$ are the state-dependent instantaneous volatility and drift rate, $\mu \in \mathbb{R}$ is a constant, and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion. We assume that $\sigma(x) > 0$ and $k(x) > 0$ are Lipschitz continuous on the interval $[\varepsilon, \infty)$ for each $\varepsilon > 0$ (i.e., locally Lipschitz), and that $\sigma(x)$ and $k(x)$ remain bounded as $x \to \infty$. We do not assume that $\sigma(x)$ and $k(x)$ remain bounded as $x \to 0$. Under these assumptions the process X does not explode to infinity (i.e., infinity is a *natural boundary* for the diffusion process; see Borodin and Salminen (2002) p.14 for boundary classification of diffusion processes). We also assume that zero is either an (unattainable) natural boundary or an entrance boundary. If zero is a natural boundary the state space is given by $E = (0, \infty)$. If zero is an entrance boundary, i.e., the process X can be started at $x = 0$ then it quickly moves to interior of $[0, \infty)$ to never hit zero again. Throughout this document we assume that the process always starts from a positive value $X_0 = x > 0$, and hence the state

space is also defined as $E = (0, \infty)$. Under all our previous assumptions X is the unique strong solution to the SDE (2.2). The transition function $P_t^0(x, dy) = \mathbb{P}(X_t \in dy | X_0 = x)$ of the diffusion process X started at x defines a Feller semigroup $\mathcal{P}^0 = \{\mathcal{P}_t^0, t \geq 0\}$ acting on the space $C([0, \infty])$ of functions continuous on $(0, \infty)$ and such that the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to\infty} f(x)$ exist and are finite (Ethier and Kurtz (1986) p.366) by

$$\mathcal{P}_t^0 f(x) = \mathbb{E}_x\big[f(X_t)\big] = \int_E f(y) P_t^0(x, dy).$$

The infinitesimal generator of $\mathcal{P}^0$ is a second-order differential operator of the form

$$\mathcal{A}^0 f(x) = \frac{1}{2} a^2(x) \frac{d^2 f}{dx^2}(x) + b(x) \frac{df}{dx}(x)$$

with the domain $\mathrm{Dom}(\mathcal{A}^0) = \{f \in C([0, \infty]) \cap C^2((0, \infty)), \mathcal{A}^0 f \in C([0, \infty])\}$ if zero is an inaccessible boundary (natural or entrance). We also note that the semigroup leaves the space $C_0((0, \infty)) \subset C([0, \infty])$ of functions continuous on $(0, \infty)$ and having zero limits $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$ invariant and is a Feller semigroup on it. Lastly, we denote by $\mathbb{F}^X = \{\mathcal{F}_t^X, t \geq 0\}$ the completed natural filtration of the process X.

The indicator process D. Let $\mathcal{E} \sim \mathrm{Exp}(1)$ be an exponential random variable with unit mean, independent of X, and define the *trigger event time* ζ as $\zeta = \inf\big\{t \geq 0 : \int_0^t k(X_s) ds \geq \mathcal{E}\big\}$. That is, ζ is the *lifetime* of the process X, defined as the first jump time of a doubly stochastic Poisson process whose jump intensity given by the killing rate $k(x)$. Observe that the killing rate is added in the drift (2.2) to compensate for the killing (jump-to-default). This compensation ensures that the stock price S is a martingale. We denote by $\mathcal{P}^1 = \{\mathcal{P}_t^1, t \geq 0\}$ the Feynman-Kac semigroup associated with the killing rate $k(X)$:

$$\mathcal{P}_t^1 f(x) = \mathbb{E}_x\big[e^{-\int_0^t k(X_u) du} f(X_t)\big] = \int_E f(y) P_t^1(x, dy), \quad t \geq 0. \tag{2.3}$$

$\mathcal{P}^1$ is a (sub-Markovian) Feller semigroup on $C([0, \infty])$ with the generator

$$\mathcal{A}^1 f(x) = \mathcal{A}^0 f(x) - k(x) f(x)$$

with the domain $\mathrm{Dom}(\mathcal{A}^1) \subseteq \mathrm{Dom}(\mathcal{A}^0)$. More precisely, $\mathrm{Dom}(\mathcal{A}^1) = \{f \in C([0, \infty]) \cap C^2((0, \infty)), \mathcal{A}^1 f \in C([0, \infty])\}$. Since zero is an inaccessible boundary (natural or entrance) of the *diffusion with killing at the rate* $k(x)$ it suffices to restrict the domain to $\mathrm{Dom}(\mathcal{A}^1) = \{f \in C_0([0, \infty]) \cap C^2((0, \infty)), \mathcal{A}^1 f \in C_0([0, \infty])\}$ whenever we start the process from the interior of E, i.e., $x > 0$ (cf. Borodin and Salminen (2002) pp.15ff).

Since the random variable ζ is not $\mathcal{F}_\infty^X$-measurable, we introduce an *indicator process* $D = \{D_t, t \geq 0\}$ in order to keep track of the event $\{\zeta \leq t\}$. The indicator process D is defined by $D_t = \mathbb{I}_{\{\zeta \leq t\}}$. Lastly, we denote by $\mathbb{F}^D = \{\mathcal{F}_t^D, t \geq 0\}$ the (completed) natural filtration of the process D.

The Lévy subordinator T. A Lévy subordinator $T = \{T_t, t \geq 0\}$ is a Lévy process with positive jumps and non-negative drift. For a standard reference on subordinators we refer the reader to Bertoin (2004). We require that T be independent of X and $\mathcal{E}$ and satisfy $T_0 = 0$. Every Lévy subordinator has the *Itô-Lévy decomposition* $dT_t = \gamma\, dt + \int_{(0,\infty)} s\, N_t(ds)$, where $\gamma \geq 0$ is the *drift* of the subordinator and N is a *Poisson random measure* with the property that, for any Borel set $A \in \mathcal{B}((0, \infty))$ we have $\mathbb{E}\left[dN_t(A)\right] = \nu(A)dt$ for some σ-finite measure ν on $\mathcal{B}((0, \infty))$. The measure ν, which must satisfy $\int_0^\infty (s \wedge 1)\, \nu(ds) < \infty$, is referred to as the *Lévy measure*. The Laplace transform of a Lévy subordinator T is given by

$$\mathbb{E}\left[e^{\lambda T_t}\right] = \int_0^\infty e^{\lambda s} \pi_t(ds) = e^{-t\phi(-\lambda)}, \qquad \lambda \in \Big\{\lambda \in \mathbb{R} : \int_{[1,\infty)} e^{\lambda s} \nu(ds) < \infty\Big\} =: \mathcal{I}, \tag{2.4}$$

where $\pi_t(ds)$ is the transition function of the subordinator T and $\phi(\lambda)$ is the *Laplace exponent* of T, which can be computed explicitly from the *Lévy-Kintchine formula*

$$\phi(\lambda) = \gamma\, \lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda s}\right) \nu(ds). \tag{2.5}$$

Note that the Laplace exponent $\phi(\lambda)$ is concave and increasing and satisfies $\phi(0) = 0$ (see Bertoin (1996), page 73). We also define $L = \{L_t, t \geq 0\}$ as the *first passage process* or *right inverse process* of T, i.e., $L_t := \inf\{s : T_s > t\}$. Recall that T is assumed to be *independent* of $\mathcal{F}_\infty^{X,D}$. Therefore, L is independent of $\mathcal{F}_\infty^{X,D}$ as well. We let $\mathbb{F}^L = \{\mathcal{F}_t^L, t \geq 0\}$ be the natural filtration of the process L.

The auxiliary bivariate process (X, D). The auxiliary bivariate process $(X, D) = \{(X_t, D_t), t \geq 0\}$ with state space $\tilde{E} = E \times \{0, 1\}$ is a Feller semimartingale in the enlarged filtration $\mathbb{F}^{X,D} = \{\mathcal{F}_t^{X,D}, t \geq 0\}$ with $\mathcal{F}_t^{X,D} = \mathcal{F}_t^X \vee \mathcal{F}_t^D$. Observe that $\mathbb{F}^{X,D}$ is the smallest filtration that contains $\mathbb{F}^X$ and in which ζ is a stopping time. In addition, observe that any function $f(x, d) \in C([0, \infty] \times \{0, 1\})$ is represented as,

$$f(x, d) = u(x)(1 - d) + v(x)d = (u - v)(x)(1 - d) + v(x), \qquad u, v \in C([0, \infty]). \tag{2.6}$$

As we shall see below, the function $u(x) = f(x, 0)$ can be interpreted as a promised payoff function if the triggering event ζ *does not* occur by time $t > 0$, while $v(x) = f(x, 1)$ can be understood as a recovery payoff function if the triggering event occurs prior time $t > 0$.

The subordinate bivariate process (X^ϕ, D^ϕ). The subordinate bivariate process is constructed by time-changing the bivariate process (X, D) with the subordinator T, i.e., $(X^\phi, D^\phi) := \{(X_{T_t}, D_{T_t}), t \geq 0\}$. This transformation is called *Bochner's subordination* due to work on subordination of semigroups and their generators by Bochner (1949). Define the filtration $\mathbb{F}^G = \{\mathcal{F}_t^G, t \geq 0\}$, with $\mathcal{F}_t^G = \mathcal{F}_t^{X,D} \vee \mathcal{F}_t^L$, to be the smallest filtration that contains $\mathbb{F}^{X,D}$ and in which T is an increasing family of $\mathbb{F}^G$-stopping times. Then the *subordinate filtration* $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is constructed by time-changing the filtration $\mathbb{F}^G$ with the Lévy subordinator T, i.e., $\mathcal{F}_t = \mathcal{F}_{T_t}^G$. The subordinate bivariate process (X^ϕ, D^ϕ) is a Feller $\mathbb{F}$-semimartingale. In credit-equity applications, the process D^ϕ is identified with the *default indicator process*, while the *default time* ζ^ϕ is defined by its right inverse, i.e., $\zeta^\phi = \inf\{t : T_t \geq \zeta\}$ where ζ is the trigger event time defied above. Observe that since the identity $\{\zeta^\phi \leq t\} = \{T_t \geq \zeta\}$ holds (see Section 5.1.1 in Jeanblanc et al. (2009)), then $D_t^\phi = \mathbb{I}_{\{T_t \geq \zeta\}} = \mathbb{I}_{\{\zeta^\phi \leq t\}}$.

Under our assumptions with respect to the diffusion coefficient $a(x)$, drift $b(x)$ and killing rate $k(x)$, the semigroups have densities with respect to the Lebesgue measure. For $\beta = 0$, the transition kernel is conservative, i.e., $P_t^0(x, E) = \int_E p^0(t, x, y)dy = 1$. For $\beta = 1$ the kernel is generally defective, i.e., $P_t^1(x, E) = \int_E p^1(t, x, y)dy \leq 1$. While our diffusion is non-negative, for future convenience we extend the transition densities from E to $\mathbb{R}$ by setting $p^\beta(t, x, y) \equiv 0$ for all $y \notin E$ and for all $x \in E$ and $t \geq 0$. From Mendoza-Arriaga and Linetsky (2014b), it then follows that the bi-variate process (X^ϕ, D^ϕ) is a Feller process with the Feller semigroup $\{\mathcal{P}_t^\phi, t \geq 0\}$ acting on $f \in C([0, \infty] \times \{0, 1\})$ by:

$$\mathcal{P}_t^\phi f(x, d) = \mathcal{P}_t^{\phi,0} v(x) + (1 - d)\mathcal{P}_t^{\phi,1}(u - v)(x),$$

where $u(x) = f(x, 0) \in C([0, \infty])$, $v(x) = f(x, 1) \in C([0, \infty])$, and $\{\mathcal{P}_t^{\phi,0}, t \geq 0\}$. The semigroups $\{\mathcal{P}_t^{\phi,\beta}, t \geq 0\}$, $\beta \in \{0, 1\}$, are Feller semigroups obtained by subordination in the sense of Bochner (1949) from Feller semigroups $\{\mathcal{P}_t^\beta, t \geq 0\}$. That is,

$$\mathcal{P}_t^{\phi,\beta} f(x) = \int_{[0,\infty)} \mathcal{P}_s^\beta f(x)\pi_t(ds), \quad t \geq 0,\ f \in C([0, \infty]),\ \beta \in \{0, 1\}, \tag{2.7}$$

where the infinitesimal generator $\mathcal{A}^\phi$ of the Feller semigroup $\{\mathcal{P}_t^\phi, t \geq 0\}$ has the following representation:

$$\mathcal{A}^\phi f(x, d) = \mathcal{A}^{\phi,0} v(x) + (1 - d)\mathcal{A}^{\phi,1}(u - v)(x), \quad u, v \in \mathrm{Dom}(\mathcal{A}^1),$$

where $\mathcal{A}^{\phi,\beta}$, $\beta \in \{0, 1\}$, are generators of $\{\mathcal{P}_t^{\phi,\beta}, t \geq 0\}$. Using Phillips (1952) theorem, Mendoza-Arriaga and Linetsky (2014b) obtained the following Lévy-Khintchine-type representation with state-dependent coefficients of the generators $\mathcal{A}^{\phi,\beta}$, $\beta \in \{0, 1\}$:

$$\begin{aligned}\mathcal{A}^{\phi,\beta} f(x) = &\frac{1}{2}\gamma a^2(x) f''(x) + b^{\phi,\beta}(x) f'(x) - k^\phi(x) f(x) \\ &+ \int_{\mathbb{R}} \Big(f(x + y) - f(x) - \mathbb{I}_{\{|y| \leq 1\}} y f'(x)\Big)\pi^{\phi,\beta}(x, y)dy, \quad f \in \mathrm{Dom}(\mathcal{A}^\beta),\end{aligned} \tag{2.8}$$

where the state-dependent Lévy density $\pi^{\phi,\beta}(x,y)$ is defined for all $y \neq x$ by

$$\pi^{\phi,\beta}(x,y) = \int_{(0,\infty)} p^{\beta}(s,x,x+y)\nu(ds), \tag{2.9}$$

and satisfies the integrability condition $\int_{\mathbb{R}}(|y|^2 \wedge 1)\pi^{\phi,\beta}(x,y)dy < \infty$ for each $x \in E$ (recall that we extended $p^{\beta}(t,x,y)$ to $\mathbb{R}$ by setting $p^{\beta}(t,x,y) \equiv 0$ for $y \notin E$), the drift with respect to the truncation function $h^{X^{\phi}}(x) = x\mathbb{I}_{\{|x|\leq 1\}}$ is given by,

$$b^{\phi,\beta}(x) = \gamma b(x) + \int_{(0,\infty)} \Big(\int_{\{|y|\leq 1\}} y p^{\beta}(s,x,x+y)dy \Big)\nu(ds),$$

and the killing rate is given by

$$k^{\phi}(x) = \gamma\beta k(x) + \int_{(0,\infty)} \Big(1 - P_s^{\beta}(x,E)\Big)\nu(ds),$$

where $P_s^{\beta}(x,E) = \int_E p^{\beta}(s,x,y)dy$. Observe that, under our conservativeness assumptions, if $\beta = 0$ then the killing rate is zero since $P_s^0(x,E) = 1$ for all $s \geq 0$.

Moreover, if $f(x,d) \in \mathrm{Dom}(\mathcal{A})$ (i.e., f is of the form (2.6) with $u, v \in \mathrm{Dom}(\mathcal{A}^1)$) and (X^{ϕ}, D^{ϕ}) starts with $X_0^{\phi} = x > 0$ and $D_0^{\phi} = d \in \{0,1\}$, then the process $M_t^f := f(X_t^{\phi}, D_t^{\phi}) - f(x,d) - \int_0^t \mathcal{A}^{\phi} f(X_s^{\phi}, D_s^{\phi})ds$ is an $\mathbb{F}$-martingale. In particular, the Doob-Meyer decomposition of D_t^{ϕ} is:

$$D_t^{\phi} = B_t^{D^{\phi}} + M_t^{\phi}, \tag{2.10}$$

where $B_t^{D^{\phi}} = \int_0^t (1 - D_s^{\phi})k^{\phi}(X_s^{\phi})ds$ is the predictable compensator of D_t^{ϕ}, and $M_t^{\phi} = D_t^{\phi} - B_t^{D^{\phi}}$ is the corresponding martingale. In the credit-equity context, the $\mathbb{F}$-intensity $\lambda_t^{\phi} = (1 - D_t^{\phi})k^{\phi}(X_t^{\phi})$ corresponds to the *default intensity process*. Due to this semimartingale characterization of the process (X^{ϕ}, D^{ϕ}) we can now obtain the characterization of the stock price process S.

From Eq. (2.1) we observe that the dynamics of the stock price S can be described by means of the subordinate bivariate process (X^{ϕ}, D^{ϕ}). Indeed, the stock price S can be seen as a function $f(t, X_t^{\phi}, D_t^{\phi}) \in C([0,\infty) \times E \times \{0,1\})$ which is decomposed according to (2.6) with the payoff function $u(t,x) = e^{\rho t}x$ if no default occurs by time $t \geq 0$, and zero recovery $v(t,x) = 0$ if the firm defaults prior to time $t \geq 0$. That is, $S_t = e^{\rho t}X_t^{\phi}(1 - D_t^{\phi})$. The scaling constant ρ is introduced to ensure that the asset S is an $\mathbb{F}$-martingale. As shown in Mendoza-Arriaga et al. (2010), S will be a martingale if and only if $\rho = \phi(-\mu)$ and $\mu \in \mathcal{I}$, where the set $\mathcal{I}$ is defined in Eq. (2.4). From the previous condition in ρ we are free to choose any value of μ as long as $\mu \in \mathcal{I}$. Hence, from this point onward we assume that $\mu \in \mathcal{I}$. Observe that the underlying assumption $v(x) = 0$ is equivalent to modeling the stock price S under *absolute priority*, which means that the stock holder has zero recovery in the event of default.

Note that Mendoza-Arriaga et al. (2010) simply define the stock price S by (2.1), but do not provide its canonical representation. The following Theorem, which is the main result of this section, provides the canonical representation of S. This result will be needed for the quadratic variation computations we perform in Section 3.

Theorem 2.1. (Canonical representation of S) *Let the stock price process S be specified in terms of the bivariate process (X^{ϕ}, D^{ϕ}) by the prescription $S_t = e^{\rho t}(1 - D_t^{\phi})X_t^{\phi}$, where $X_0^{\phi} = x > 0$ and $D_0^{\phi} = d \in \{0,1\}$. Moreover, assume that the scaling factor ρ satisfies $\rho = \phi(-\mu)$ where $\phi(u)$ is the Laplace exponent (2.5) of the subordinator T, and where $\mu \in \mathcal{I}$ is the constant drift of the background process X (2.2) (the set $\mathcal{I}$ is defined in Eq. (2.4)). Then, the stock price process S is a martingale with canonical representation*

$$S_t = S_0 + \sqrt{\gamma}\int_0^t \sigma(X_u^{\phi})S_u dW_u + \int_0^t \int_{\mathbb{R}} e^{\rho s}(1 - D_{s-}^{\phi})y\left(\hat{\mu}(ds,dy) - \pi^{\phi,1}(X_{s-}^{\phi}, y)dyds\right) - \int_0^t S_{u-}dM_u^{\phi},$$

where $\gamma \geq 0$ is the drift of the Lévy subordinator T. Here, the random measure

$$\hat{\mu}(\omega; ds, dy) = \sum_u \mathbb{I}_{\{\Delta X_u^{\phi}(\omega)\neq 0\}}\mathbb{I}_{\{\Delta D_u^{\phi}(\omega)=0\}}\delta_{(u,\Delta X_u^{\phi}(\omega))}(ds,dy),$$

corresponds to those jumps of X^ϕ that do not *coincide with jump of the default indicator D^ϕ. The Lévy density $\pi^{\phi,1}(x,y)$ is defined in Eq.* (2.9)*. W is a Brownian motion and M^ϕ is the martingale* (2.10) *associated to D^ϕ.*

Proof. From the restrictions on μ and ρ, the stock price S_t is a martingale and can be decomposed as $S_t = S_0 + \int_0^t S_{u-} dM_u$, where M_t is a special martingale. Then, the canonical representation of S follows from the Itô formula of Mendoza-Arriaga and Linetsky (2014b), Corollary 3.3, applied to the function $f(t,x,d)$ defined by $u(t,x) = e^{\rho t}x$ and $v(t,x) = 0$. □

This canonical representation decomposes the stock price process S into a purely discontinuous martingale of jumps prior to default with the compensator measure $(1 - D^\phi_{u-})\pi^{1,\phi}(X^\phi_{u-}, y)dydu$, a continuous martingale component represented in terms of a Brownian motion, and a final jump to zero (the default term $-\int_0^t S_{u-} dM^\phi_u$). Clearly, the process S is a killed jump-diffusion process whenever $\gamma > 0$, and a purely discontinuous process for $\gamma = 0$.

3 Variance swap computation

With the semimartingale characterization of the stock price process S at our disposal, we are now in position to compute the value of variance swap rate $K_{\text{var}} = \mathbb{E}([\log S]_t)$. First, observe that if the firm underlying S were to default at some time ζ^ϕ in the interval $[0,t]$, then the payoff of a traditional variance swap contract (1.1) is not well defined. To account for this possibility, we modify the floating leg of the VS so that it only accumulates quadratic variation *prior* to the default time ζ^ϕ. That is, the long side of a VS, under our modified definition, has a payoff of

$$[\log S]_{t\wedge\zeta^\phi -} - K_{\text{var}} = \int_0^t (1 - D^\phi_u) d[\log S]_u - K_{\text{var}}. \tag{3.1}$$

Notice that, for an asset that cannot default, our modified definition of a VS coincides with the traditional definition of a VS. Meanwhile, for an asset that *can* default, our modified definition of a VS is guaranteed to have a finite payoff, since the floating leg of the modified VS only accumulates quadratic variation up the time just prior to default.[1]

Remark 3.1. Carr and Schoutens (2008) also consider VSs written on defaultable assets. They define the floating leg of a VS as the quadratic variation of relative returns up to (and including) a possible jump to zero. Thus, in their framework, the floating leg of the VS has a payoff of

$$[Y]_{t\wedge\zeta^\phi}, \qquad \text{where} \qquad dY_t = \frac{dS_t}{S_{t-}}.$$

Note that, when a default occurs, the floating leg of the VS jumps by +1. After a default, the floating leg does not accumulate quadratic variation. The advantage our modification of the VS contract is that it separates default risk from volatility risk, whereas the definition of Carr and Schoutens (2008) does not separate these two.

Using definition (3.1), the fair value of K_{var} is the risk-neutral expectation of the floating leg

$$K_{\text{var}} = \mathbb{E}_x \left[\int_0^t (1 - D^\phi_u) d[\log S]_u \right]. \tag{3.2}$$

An explicit expression for the right-hand side of (3.2) is given in the following theorem.

Theorem 3.2. *Let S be given by $S_t = e^{\rho t} X^\phi_t (1 - D^\phi_t)$. Then the right-hand side of* (3.2) *is given by*

$$K_{var} = \int_0^t \mathbb{E}_x \left[\mathbb{I}_{\{\zeta^\phi > u\}} \gamma \sigma^2(X^\phi_u) \right] du + \int_0^t \mathbb{E}_x \left[\mathbb{I}_{\{\zeta^\phi > u\}} \int_{\mathbb{R}} \log^2 \left(1 + \frac{y}{X^\phi_{u-}} \right) \pi^{\phi,1}(X^\phi_{u-}, y) dy \right] du. \tag{3.3}$$

[1] Note that market practice is to cap the realized variance of the contract by 2.5 times the VS rate. This cap would automatically be attained in the case of a default. Thus, in this case, variance risk is intimately tied with default risk. Here, what we propose is an alternative definition of the VS contract, which separates variance risk from default risk.

Proof. In view of Eq. (3.2), it suffices to first calculate the Lévy-Itô canonical representation of the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$, which corresponds to a zero–recovery log–contract on the stock price. That is, a contract that pays $u(t, X_t^\phi) = \log(e^{\rho t}X_t^\phi) = \log S_t$ if no default occurs by time $t \geq 0$, and zero otherwise (i.e., we set $v(t, X_t^\phi) = 0$). Hence, the canonical representation of the pre-default log–contract of S can be obtained by means of an application of the Itô formula of Mendoza-Arriaga and Linetsky (2014b), Corollary 3.3, to the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$,

$$\begin{aligned} d(\log S_t) =& \big[(1-D_{t-}^\phi)\big(\partial_t + \mathcal{A}^{\phi,1}\big)\log(e^{\rho t}X_{t-}^\phi)\big]\,dt + (1-D_{t-}^\phi)\,\sqrt{\gamma}\,\sigma(X_{t-}^\phi)dW_t \\ &+ (1-D_{t-}^\phi)\int_{\mathbb{R}}\Big(\log\big(e^{\rho t}(X_{t-}^\phi + y)\big) - \log\big(e^{\rho t}(X_{t-}^\phi)\big)\Big)\Big(\hat{\mu}(dt,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\Big) \\ &- (1-D_{t-}^\phi)\log(e^{\rho t}X_{t-}^\phi)dM_t^\phi. \end{aligned} \tag{3.4}$$

Observe that due to the default term $(1-D_{t-}^\phi)\log(e^{\rho t}X_{t-}^\phi)dM_t^\phi = (1-D_{t-}^\phi)\log(S_{t-})dM_t^\phi$ the process jumps to zero at default, which is consistent with our selection of the function $f(t,x,d) = (1-d)\log(e^{\rho t}x)$ that has zero-recovery in case of default. Consequently, it describes the pre-default dynamics of $\log S_t$ and prevents it from exploding at default time. From (3.4) it is straightforward to compute the differential $d[\log S]_t$ of the pre-default dynamics of $\log S_t$,

$$\begin{aligned} d[\log S]_t =& (1-D_{t-}^\phi)\left(\gamma\sigma^2(X_t^\phi) + \int_{\mathbb{R}}\log^2\Big(1+\frac{y}{X_{t-}^\phi}\Big)\pi^{\phi,1}(X_{t-}^\phi, y)dy\right)dt \\ &+ (1-D_{t-}^\phi)\int_{\mathbb{R}}\log^2\Big(1+\frac{y}{X_{t-}^\phi}\Big)\Big(\hat{\mu}(dt,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\Big) \\ &+ (1-D_{t-}^\phi)\log^2(e^{\rho t}X_{t-}^\phi)dD_t^\phi. \end{aligned} \tag{3.5}$$

Finally, multiplying (3.5) by $1-D_t^\phi = \mathbb{I}_{\{t<\zeta^\phi\}}$, observing that $(1-D_{t-}^\phi)(1-D_t^\phi) = (1-D_t^\phi)$ and $(1-D_t^\phi)dD_t^\phi = 0$ a.s., integrating over the interval $[0,t]$, taking an expectation, and using the fact that the random measure $(1-D_{t-}^\phi)\big(\hat{\mu}(dt,dy) - \pi^{\phi,1}(X_{t-}^\phi, y)dydt\big)$ is a martingale measure, one arrives at (3.3). □

4 Spectral expansions

In order for the results of Section 3 to be useful, we compute the FK semigroups $\{\mathcal{P}_t^\beta, t \geq 0\}$ and $\{\mathcal{P}_t^{\phi,\beta}, t \geq 0\}$, $\beta \in \{0,1\}$, using the spectral representation of symmetric semigroups under subordination. Our goal in this section is to concisely review the results that will be needed in subsequent sections. A more detailed exposition can be found, e.g., in Mendoza-Arriaga et al. (2010), Li and Linetsky (2013a,b).

We consider the case where the FK semigroup $\mathcal{P}_t^\beta$ of the background diffusion process X is a strongly continuous semigroup of symmetric contractions on $L^2(E,\mathfrak{m}) := \{f : \int_E f^2(x)\mathfrak{m}(dx) < \infty\}$ (where $\mathfrak{m}(dx) = \frac{2}{a^2(x)}\exp\big(\int^x \frac{2\,b(y)}{a^2(y)}dy\big)dx$ is the speed measure), which are generated by the self-adjoint dissipative second-order differential operators $\mathcal{A}^\beta$ (see McKean (1956)). Hence, $\mathcal{P}_t^\beta$ admits a spectral decomposition. Once the spectral decomposition $\mathcal{P}_t^\beta$ of the underlying diffusion process is known, Propositions 1–2 in Li and Linetsky (2013b) give us the spectral decomposition of the subordinate FK semigroup $\mathcal{P}_t^{\phi,\beta}$, thus allowing us to derive analytical formulas for financial derivatives with square-integrable payoffs under models based on additive subordinate diffusions.

We assume, for simplicity, that the spectrum of $\mathcal{A}^\beta$ is purely discrete. Many diffusion processes used in finance have purely discrete spectrum, and the explicit expressions for many examples can be found in Linetsky (2004, 2008) (the continuous spectrum case can be obtained by inverting the resolvent operator as in, e.g., Lewis (1998) and Mendoza-Arriaga et al. (2010)). We assume that for each $t > 0$, $\mathcal{P}_t^\beta$ is trace-class (see Reed and Simon (1980) p.206 for the definition). In this case, it has a symmetric kernel $p_{\mathfrak{m}}(t,x,y)$, i.e., $\mathcal{P}_t^\beta f(x) = \int_E p_{\mathfrak{m}}^\beta(t,x,y)f(y)\mathfrak{m}(y)dy$, $p_{\mathfrak{m}}^\beta(t,x,y) = p_{\mathfrak{m}}^\beta(t,y,x)$ and $p^\beta(t,x,y) = p_{\mathfrak{m}}^\beta(t,x,y)\mathfrak{m}(y)$ (Davies (2007), Lemma 7.2.1). We further assume that $p_{\mathfrak{m}}(t,x,y)$ is continuous in (x,y) for each $t > 0$. Under these assumptions, Propositions 1–2 in Li and Linetsky (2013b) show that for each $t > 0$, $\mathcal{P}_t^\beta$ can be represented

by the eigenfunction expansion

$$\mathcal{P}_t^\beta f(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\psi_n(x)c_n, \quad \text{with} \quad c_n = (f,\psi_n) = \int_E f(y)\psi_n(y)\mathfrak{m}(y)dy, \quad f \in L^2(E,\mathfrak{m}), \tag{4.1}$$

where $\psi_n(x)$ and λ_n are the eigenfunctions and eigenvalues, and c_n are the expansion coefficients. Also from Proposition 2 in Li and Linetsky (2013b), (4.1) not only converges in the L^2-sense but it converges uniformly on compacts (u.o.c.). Moreover, $\mathcal{P}_t^\beta f(x)$ is a continuous function in x for all $f \in L^2(E,\mathfrak{m})$, and the eigenfunctions satisfy $|\psi_n(x)| \leq e^{\lambda_n t/2}\sqrt{p_{\mathfrak{m}}^\beta(t,x,x)}$ for all $t > 0$.

From Bochner's subordination (see Example 12.6 and Remark 13.4 in , the subordinate FK semigroup $\mathcal{P}_t^{\phi,\beta}$ also admits an spectral representation,

$$\mathcal{P}_t^{\phi,\beta} f(x) = \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)c_n, \quad \text{with} \quad c_n = (f,\psi_n) = \int_E f(y)\psi_n(y)\mathfrak{m}(y)dy, \quad f \in L^2(E,\mathfrak{m}). \tag{4.2}$$

Observe that (4.2) differs from (4.1) only on the set of eigenvalues (i.e., for $n = 1, 2, \cdots$, λ_n are the eigenvalues of $\mathcal{P}_t^\beta$ while $\phi(\lambda_n)$ are the eigenvalues of $\mathcal{P}_t^{\phi,\beta}$). Hence, given the spectral representation of $\mathcal{P}_t^\beta$ we immediately obtain the corresponding spectral representation of $\mathcal{P}_t^{\phi,\beta}$ as long as the Lévy-Kintchine exponent $\phi(\lambda)$ is available. Also observe that the spectral representation of $\mathcal{P}_t^{\phi,\beta}$ (4.2) converges, in general, only in the L^2-sense, even if (4.1) converges o.u.c. However, in financial applications, it is preferable to have (4.2) converging u.o.c., since we are interested in derivative prices at particular values of the underlying variable in a compact domain, and L^2-sense convergence does not guarantee convergence at a given point. Fortunately, Proposition 1 in Li and Linetsky (2013a) provides us with sufficient conditions for (4.2) to converge u.o.c. Basically, if the subordinator T has a (positive) drift $\gamma > 0$ then (4.2) automatically converges u.o.c., on the other hand, if $\gamma = 0$ the condition of Proposition 1 in Li and Linetsky (2013a) is mild. Our main example, the subordinate JDCEV process, satisfies these conditions.

In order to make use of the spectral expansions machinery, we have the following lemma, which provides an alternative formulation of the value of K_{var} in terms of Feynman-Kac semigroups.

Lemma 4.1. *Let S be given by $S_t = e^{\rho t}X_t^\phi(1 - D_t^\phi)$ with $X_0 = x > 0$ and $D_0^\phi = d \in \{0,1\}$. Also, let $\mathcal{P}_t^1$ (resp., $\mathcal{P}_t^{\phi,1}$) be the (resp., the subordinate) Feynman-Kac semigroup defined in Eq.* (2.3) *(resp., Eq.* (2.7)*). Then, K_{var} can be represented as follows,*

$$K_{var} = \mathbb{I}_{\{\zeta^\phi > 0\}}\,\gamma \int_0^t (\mathcal{P}_u^{\phi,1}\sigma^2(x))du + \mathbb{I}_{\{\zeta^\phi > 0\}} \int_0^t \Big(\int_{(0,\infty)} \mathcal{P}_u^{\phi,1} f(s,\cdot)(x)\nu(ds)\Big)du, \tag{4.3}$$

with

$$f(s,y) = \mathcal{P}_s^1 \log^2(y) - 2\log(y)\mathcal{P}_s^1 \log(y) + \log^2(y)\mathcal{P}_s^1 1, \quad y = X_u^\phi. \tag{4.4}$$

Proof. See Appendix B □

In summary, the value of a variance swap rate K_{var} is obtained by applying the spectral expansions (4.1)-(4.2) to the functions of equation (4.3)-(4.4). In section 5 we present the explicit calculations for the subordinate JDCEV specification.

Remark 4.2. While we limit the analysis in this paper to a stock S whose background Markov processes X is a diffusion, the mathematical machinery we develop can be extended to any Markov process X whose generator has a known eigenfunction expansion. For instance, Li et al. (2015) recently obtained an analytical representation of the transition semigroup of the Basic Affine Jump diffusion (BAJD) process, which resembles an eigenfunction expansion and for which subordination can be applied in the same way as proposed here. Hence, the BAJD could be a jump-diffusion alternative for the background process X. Nonetheless, even restricting X to diffusion processes as we do here, the model for S is quite flexible. As such, we do not pursue the possibility of allowing X to be more general here.

5 Example: the subordinate JDCEV process

In this Section, we compute K_{var} (3.3) explicitly (up to an integral with respect to the Lévy measure ν of the subordinator T), when the diffusion X is modeled as a *Jump-to-default Constant Elasticity of Variance (JDCEV) process*. The JDCEV process was introduced by Carr and Linetsky (2006) to extend the well-known CEV model of Cox (1975) in order to include a possible jump-to-default. In the JDCEV framework, the jump to default has a killing rate which is an affine functions of the local variance, that is

$$\sigma(x) = ax^{\beta}, \quad \text{and} \quad k(x) = b + c\sigma^2(x) = b + ca^2x^{2\beta},$$

where $b \geq 0$ and $c \geq 0$. Here, $\beta < 0$ is the volatility elasticity parameter and $a > 0$ is the volatility scale parameter. The specification $\beta < 0$ is consistent with the leverage effect (volatility increases when the stock price falls). Similarly, from the credit risk perspective, the probability of default increases (the killing rate increases) as the stock price falls. When $c \geq 1/2$ the zero boundary is *entrance* for the JDCEV diffusion, and thus, the diffusion cannot reach zero from the interior of E. Nonetheless, for all $c > 0$ default may only occur through a jump from a positive value. The majority of the expressions developed in this Section hold for all $c > 0$.

The spectral representation of the JDCEV semigroup is well known (see, e.g., Mendoza-Arriaga and Linetsky (2014a)). The speed density of the JDCEV process is given by:

$$\mathfrak{m}(x) = \frac{2}{a^2}x^{2c-2-2\beta}e^{\varepsilon Ax^{-2\beta}}, \quad \varepsilon := \text{sign}(\mu + b).$$

When $\mu+b \neq 0$[2], the JDCEV semigroup is a symmetric trace-class semigroup in $L^2(E, \mathfrak{m})$ with the eigenvalues and continuous eigenfunctions of the negative of its infinitesimal generator $\mathcal{A}^1$ defined in $L^2(E, \mathfrak{m})$, and with the Dirichlet (vanishing) boundary condition at 0):

$$\lambda_n = \omega(n-1) + \lambda_1, \;\; \psi_n(x) = A^{\frac{\nu}{2}}\sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x\, e^{-\frac{1}{2}(1+\varepsilon)Ax^{-2\beta}} L_{n-1}^{\nu}(Ax^{-2\beta}), \;\; n = 1, 2, \ldots,$$

where we introduced the notation

$$\nu := \frac{1+2c}{2|\beta|}, \quad \omega := 2|\beta(\mu+b)|, \quad A := \frac{|\mu+b|}{a^2|\beta|}, \quad \lambda_1 := \begin{cases} 2(\mu+b)(|\beta|+c)+b, & \mu+b > 0 \\ |\mu|, & \mu+b < 0 \end{cases}.$$

λ_1 is called the principal eigenvalue, and $L_n^{\nu}(x)$ are the generalized Laguerre polynomials. We observe that on each compact interval $K \subset E$ there exists a constant C_K independent of n such that $|\psi_n(x)| \leq C_K n^{-1/4}$ for all $n \geq 1$ (see inequality (27a) on p.54 of Nikiforov and Uvarov (1988)). Therefore, the eigenfunctions $\psi_n(x)$ satisfy the estimates of Proposition 2 in Li and Linetsky (2013b), and hence, for all $t > 0$ the spectral expansion of $\mathcal{P}_t^{\phi,1}$ (as well as of $\mathcal{P}_t^1$) converges u.o.c. for all $f \in L^2(E, \mathfrak{m})$. We notice, however, that in propositions 5.1-5.3 we will also need to investigate the type of convergence at $t = 0$ since to obtain K_{var} we need to integrate the FK semigroup with respect to time, i.e., $\int_0^t \mathcal{P}_u^{\phi,1} f(x) du$.

To save space, in the rest of this section we restrict our attention to the case with $\mu + b < 0$. Thus in this case, it can be easily verified that,

$$\begin{aligned}
\partial_x\psi_n(x) &= A^{\frac{\nu}{2}}\sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\Big(L_{n-1}^{\nu}(Ax^{-2\beta}) - 2|\beta|\, Ax^{-2\beta}\, L_{n-2}^{\nu+1}(Ax^{-2\beta})\Big), \\
\partial_{xx}\psi_n(x) &= 2|\beta|A^{\frac{\nu}{2}+1}\sqrt{\frac{(n-1)!|\mu+b|}{\Gamma(\nu+n)}}\, x^{-2\beta-1}\Big(2|\beta|\, Ax^{-2\beta}\, L_{n-3}^{\nu+2}(Ax^{-2\beta}) - (1+2|\beta|)L_{n-2}^{\nu+1}(Ax^{-2\beta})\Big)
\end{aligned} \tag{5.1}$$

Moreover, since $L_k^{\nu}(0) = \frac{\Gamma(k+\nu+1)}{\Gamma(k+1)\Gamma(\nu+1)}$ then have

$$\psi_n(0) = 0 \quad \textit{and} \quad \partial_x\psi_n(0) = \frac{A^{\frac{\nu}{2}}}{\Gamma(\nu+1)}\sqrt{\frac{\Gamma(\nu+n)|\mu+b|}{(n-1)!}}. \tag{5.2}$$

[2]When $\mu + b = 0$, the spectrum is purely absolutely continuous. The spectral representation of the transition density in this case with continuous spectrum is given in Mendoza-Arriaga et al. (2010).

Next, using theorem 2.1, the stock price process S is specified using a subordinate killed JDCEV process $(1-D_t^\phi)X_t^\phi = (1-D_{T_t})X_{T_t}$, where $(1-D)X$ is a killed JDCEV process and T is a Lévy subordinator with Lévy-Kintchine exponent ϕ to be determined by the market. Under this specification the value of a European call option is given as (again, recall that we have assumed zero interest rate, zero dividends, and $\mu + b < 0$)

$$
\begin{aligned}
C(t,x,K) =& \mathbb{E}[(S_t-K)^+] = x + \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)\big(p_n(K) - Kd_n\big) \quad \text{with} \qquad (5.3)\\
p_n(K) =& KA^{\frac{\nu}{2}+1}\tilde{K}^{2(c-\beta)}\sqrt{\frac{\Gamma(\nu+n)}{(n-1)!|\mu+b|}}\times \\
& \left\{\Gamma\Big(1+\frac{c}{|\beta|}\Big)\,{}_2\tilde{F}_2\Big(\begin{array}{ll}\nu+n, & 1+\frac{c}{|\beta|}\\ \nu+1, & 2+\frac{c}{|\beta|}\end{array};-A\tilde{K}^{-2\beta}\Big) - {}_1\tilde{F}_1\Big(\begin{array}{l}\nu+n\\ \nu+2\end{array};-A\tilde{K}^{-2\beta}\Big)\right\}\\
d_n =& \frac{A^{\frac{1-2c}{4|\beta|}}\,(1/(2|\beta|))_{n-1}\,\Gamma(c/|\beta|+1)}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}},
\end{aligned}
$$

where the modified strike is given by $\tilde{K}_t = Ke^{-\rho t}$, $(z)_n = z(z-1)...(z-n-1)$ is the Pochhammer symbol, and where ${}_1\tilde{F}_1$ and ${}_2\tilde{F}_2$ are the regularized Kummer confluent hypergeometric function and the regularized generalized hypergeometric function, respectively. The spectral representation (5.3) of the call option is calculated via put-call parity. We recall that, in the credit-equity framework, a put contract has two components, $P(t,x,K) = P_0(t,x,K) + P_D(t,x,K)$ (see, e.g., Carr and Linetsky (2006) and Mendoza-Arriaga et al. (2010) for details). P_0 corresponds to the portion of the contract that pays $(K-S_t)^+$ at maturity $t>0$ if the firm does not default by time t. P_D corresponds to the portion of the contract that pays K dollars at maturity if the firm defaults by time t. The coefficients $p_n(K) = e^{\rho t}\big((\tilde{K}-x)^+,\psi_n\big)$ and $d_n = \big(1,\psi_n\big)$ are computed using the integrals 2.19.2.6 and 2.19.2.3 in Prudnikov et al. (1986), respectively.

Before computing the expectations necessary to obtain K_{var} it will be useful to give an analytical solution for the p-th moment of the stock price

$$
\mathbb{E}_x\Big[(S_t)^p\Big] = e^{p\rho t}\,\mathbb{E}_x\Big[(X_t^\phi)^p\mathbb{I}_{\{\zeta^\phi>t\}}\Big] = e^{p\rho t}\,\mathcal{P}_t^{\phi,1}x^p.
$$

Proposition 5.1 (p-th Moment)**.** *Assume $\mu+b<0$. For $p>2(\beta-c)$, the expected value of the function $f(x)=x^p$ is given by the eigenfunction expansion:*

$$
\mathbb{E}_x\Big[(S_t)^p\Big] = e^{p\rho t}\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\tilde{c}_n\psi_n(x),
$$

where $\phi(\lambda)$ is the Laplace exponent of the subordinator T. The expansion coefficients are given by:

$$
\tilde{c}_n = (x^p,\psi_n) = \frac{A^{\frac{\nu}{2}-\frac{p+2c}{2|\beta|}}\left(\frac{1-p}{2|\beta|}\right)_{n-1}}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}}\Gamma\left(\frac{p+2c}{2|\beta|}+1\right), \quad n=1,2,\cdots. \qquad (5.4)
$$

Also, the spectral expansion is uniformly convergent for all $t>0$, absolutely convergent at $t=0$, and uniformly convergent at $t=0$ if $p>(\beta+1)/2-c$.

Proof. We observe that we need to compute $c_n^p = \int_0^\infty y^p\varphi_n(y)\mathfrak{m}(y)dy$. By applying the change of variable $x=y^{-2\beta}$ the integral is reduced to the formula 2.19.3.5 in Prudnikov et al. (1986), pp. 463 (we commit the details). In Appendix B.2 we show that the spectral expansion of $\mathcal{P}_t^{\phi,1}x^p$ also converges absolutely at $t=0$. In addition, if $p>(\beta+1)/2-c$, then the spectral expansion converges uniformly at $t=0$. □

We are now ready to compute K_{var}. In the following propositions, we calculate the two expectations in (3.3). These results are based on Lemma 4.1, which gives an alternative representation of K_{var} in terms of Feynman-Kac semigroups.

Proposition 5.2. *Assume $\mu + b < 0$. Then we have*

$$\gamma \int_0^t \mathbb{E}_x\Big[\mathbb{I}_{\{\zeta^\phi > u\}} \sigma^2(X_u^\phi)\Big] du$$
$$= \gamma a^2 A^{\frac{\nu}{2} - \frac{c}{|\beta|} + 1} \Gamma\left(c/|\beta|\right) \sum_{n=1}^{\infty} \frac{(1/(2|\beta|) + 1)_{n-1}}{\sqrt{(n-1)!|\mu + b|\Gamma(\nu + n)}} \frac{\big(1 - e^{-\phi(\lambda_n)t}\big)\psi_n(x)}{\phi(\lambda_n)}. \tag{5.5}$$

Proof. The expectation in (5.5) can be written explicitly as

$$\gamma a^2 \int_0^t \mathbb{E}_x\Big[\mathbb{I}_{\{\zeta^\phi > u\}} (X_u^\phi)^{2\beta}\Big] du$$
$$= \gamma a^2 A^{\frac{\nu}{2} - \frac{c}{|\beta|} + 1} \Gamma\left(c/|\beta|\right) \int_0^t \Big(\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)u} \frac{(1/(2|\beta|) + 1)_{n-1}}{\sqrt{(n-1)!|\mu + b|\Gamma(\nu + n)}} \psi_n(x) \Big) du,$$

where the last equality is due to Proposition 5.1. One should note that if $3\beta + 2c > 1$ then the sum converges uniformly at $t = 0$ and the integral can be done term by term. Otherwise, observe that: (a) the series inside the integral is absolutely convergent for all $t \geq 0$ due to Proposition 5.1 (and continuous for all $t \geq 0$), and (b) the Laplace exponent $\phi(\lambda)$ is increasing. Then, we can conclude that the resulting series (5.5) is also absolutely convergent, and hence the exchange of sum and integral is justified (i.e., we integrate term by term with $\int_\epsilon^t \cdot ds$ for some $\epsilon > 0$ and then take the limit as $\epsilon \downarrow 0$). □

Proposition 5.3. *Assume $\mu + b < 0$ and $2c - 2\beta > 1$. Then we have*

$$\int_0^t \mathbb{E}_x \left[\mathbb{I}_{\{\zeta^\phi > u\}} \int_{\mathbb{R}} \log^2\Big(1 + \frac{y}{X_{u-}^\phi}\Big) \pi^{\phi,1}(X_{u-}^\phi, y) dy \right] du$$
$$= \int_{(0,\infty)} \left\{ \sum_{n=1}^{\infty} e^{-\lambda_n s} \frac{1 - e^{-\phi(\lambda_n)t}}{\phi(\lambda_n)} \Xi_n \, \psi_n(x) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-\lambda_n s} \frac{1 - e^{-\phi(\lambda_m)t}}{\phi(\lambda_m)} \Omega_{n,m} \, \psi_m(x) \right\} \nu(ds), \tag{5.6}$$

where the coefficients Ξ_n and $\Omega_{n,m}$ are given by

$$\Xi_n = \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2} \sqrt{\frac{(n-1)!}{|\mu + b|\Gamma(\nu + n)}} \Theta_{n-1}^2\Big(\frac{c + |\beta|}{|\beta|}\Big)$$
$$\Omega_{n,m} = \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2} \sqrt{\frac{(m-1)!}{|\mu + b|\Gamma(\nu + m)}}$$
$$\times \sum_{k=0}^{n-1} \frac{(1-n)_k}{\Gamma(\nu + k + 1)\, k!} \left\{ \frac{\big(\frac{1}{2|\beta|}\big)_{n-1} \Gamma\big(\frac{c+|\beta|}{|\beta|}\big)}{(n-1)!} \Theta_{m-1}^2(\nu + k + 1) - 2\Theta_{m-1}^1(\nu + k + 1)\Theta_{n-1}^1\Big(\frac{c + |\beta|}{|\beta|}\Big) \right\}$$

with

$$\Theta_n^1(\alpha) := \begin{cases} \frac{(-1)^{m-1} m!}{n!} \Gamma(n - m)\Gamma(\nu + m + 1) & \left[\begin{array}{c} \alpha = \nu + m + 1 \\ m = 0, 1, \cdots, n-1 \end{array} \right] \\ \frac{(1-\alpha+\nu)_n}{n!} \Gamma(\alpha)\Big[\psi(\alpha) - \sum_{p=1}^n \frac{1}{(p-\alpha+\nu)}\Big], & \alpha > 0 \end{cases},$$

$$\Theta_n^2(\alpha) := \begin{cases} \frac{2(-1)^m m! \Gamma(n-m)\Gamma(\nu+m+1)}{n!} \big(\psi(n - m) - \psi(m + 1) - \psi(\nu + m + 1)\big) & \left[\begin{array}{c} \alpha = \nu + m + 1 \\ m = 0, 1, \cdots, n-1 \end{array} \right] \\ \frac{(1-\alpha+\nu)_n}{n!} \Gamma(\alpha)\Big[\Big(\psi(\alpha) - \sum_{p=1}^n \frac{1}{p-\alpha+\nu}\Big)^2 + \psi'(\alpha) - \sum_{p=1}^n \frac{1}{(p-\alpha+\nu)^2}\Big], & \alpha > 0 \end{cases},$$

and $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ (with no subscript on ψ) is the Polygamma function.

Proof. The restriction $2c - 2\beta > 1$ is imposed such that the sums converge absolutely at $t = 0$. Indeed, it is easy to show that for all $x > 0$ we have $|\log(x)| \leq (1/x + x)$ and $\log^2(x) \leq (1/x + x)$. Moreover, since the expansion of $(1/x + x)$ converges absolutely at $t = 0$ due to Proposition 5.1, then each of the series also converge absolutely at $t = 0$. The rest of the proof consists of integrating term by term. Details are found in Appendix B.3. □

Remark 5.4. Observe that for $\nu((0,\infty)) = \alpha < \infty$, i.e., T is of finite activity, the integral (5.6) can be done term by term, yielding $\int_{(0,\infty)} e^{-\lambda_n s}\nu(ds) = \big(\alpha + \gamma\lambda_n - \phi(\lambda_n)\big)$, see (6.17). For infinite activity subordinators T (i.e., $\nu((0,\infty)) = \infty$), the integral can be approximated by a finite activity subordinator T^ϵ with $\nu(\epsilon,\infty) = \alpha^\epsilon < \infty$, for a small $\epsilon > 0$ (see section 6.3 in Cont and Tankov (2004) for a discussion).

Remark 5.5. The terms of the form $(1 - e^{-\phi(\lambda_n)t})/\phi(\lambda_n)$ in (5.5) and (5.6) result from the integral $\int_\epsilon^t e^{-\phi(\lambda_n)u}du = (e^{-\phi(\lambda_n)\epsilon} - e^{-\phi(\lambda_n)t})/\phi(\lambda_n)$, evaluated at $\epsilon = 0$. However, computationally, one can improve the sums' convergence by using a small $\epsilon > 0$ instead. In particular, when $\epsilon > 0$, the sums converge uniformly. Also in this case, the restriction $2c - 2\beta > 1$ in Proposition 5.3 can be relaxed.

6 Semi-parametric calibration and market implied subordinators

A typical calibration procedure for S involves specifying parametric models for the background diffusion X and subordinator T, and calibrating the unknown model parameters to liquid European call and put options. One problem with this approach is that a fully parametric model constricts the range of processes S can follow and therefore limits the chance that model-induced option prices can match observed option prices. Here, we propose a semi-parametric calibration procedure, which provides more modeling flexibility. Specifically, we assume a parametric model for X, but we leave the subordinator T fully non-parametric. Our calibration procedure consists of two steps:

Step 1: Calibrate the parameters of X from observed European call/put prices.

Step 2: Obtain *non-parametric* representations of γ and ν, the drift and Lévy measure of T.

We now describe these steps in detail.

Step 1: Calibration of the parameters of X

Let S be described by (2.1). We assume that the diffusion coefficient $\sigma(x)$ and killing coefficient $k(x)$ of the background Feller process X are of a known parametric form, but that the values of the parameters are unknown. This information is sufficient to construct the parametric form of the eigenfunctions ψ_n and eigenvalues λ_n of $\mathcal{A}^1$. We make no assumptions about the parametric form of ν, the Lévy measure of the subordinator T, and we make no assumption about the value of γ, the drift of T.

Denote by $C(t,x;K)$ the price of a European call option with time to maturity t and strike price K. Note that the price of a call with strike price K can be obtained from the price of a put with the same strike through put-call parity. We assume the existence of t-maturity European call or put option at all strikes $K \in (0,\infty)$. [3]

Let $p_S(t,x,y)$ be the transition density of S under the risk-neutral pricing measure

$$p_S(t,x,y)dy = \mathbb{P}_x\left[S_t \in dy\right].$$

Note that $S_t \in dy$ if and only if $(1 - D_t^\phi)e^{\rho t}X_t^\phi \in dy$. Thus,

$$p_S(t,x,y) = e^{-\rho t}p^{\phi,1}(t,x,y') = e^{-\rho t}\mathfrak{m}(y')\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)\psi_n(y'), \qquad y' := y\,e^{-\rho t}. \tag{6.1}$$

As Breeden and Litzenberger (1978) show, the transition density $p_S(t,x,y)$ can be implied from a semi-infinite strip of call prices. We have

$$C(t,x;K) = \mathbb{E}_x\left[(S_t - K)^+\right] = \int_E (y-K)^+ p_S(t,x,y)dy. \tag{6.2}$$

[3] While calls at all strikes $K \in (0,\infty)$ do not trade in practice, Bondarenko (2003) shows how to estimate the value of call at any strike, given the value of calls at a discrete set of strikes.

Differentiating both sides of (6.2) twice with respect to K, and noting that $\partial^2_{KK}(y-K)^+ = \delta(y-K)$, one obtains

$$\partial^2_{KK} C(t,x;K) = \int_E \partial^2_{KK}(y-K)^+ p_S(t,x,y)dy = p_S(t,x,K). \tag{6.3}$$

Setting the two expressions (6.1) and (6.3) for p_S equal to each other yields

$$\partial^2_{KK} C(t,x;K) = e^{-\rho t}\mathfrak{m}(K') \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)\psi_n(K'), \qquad K' := K\, e^{-\rho t}. \tag{6.4}$$

Multiplying both sides of (6.4) by $\psi_{n'}(K')$ and integrating with respect to K, we obtain

$$\begin{aligned}\int_E \partial^2_{KK} C(t,x;K)\psi_{n'}(K')dK &= e^{-\rho t}\sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)\int_E \psi_n(K')\psi_{n'}(K')\mathfrak{m}(K')dK \\ &= \sum_{n=1}^{\infty} e^{-\phi(\lambda_n)t}\psi_n(x)\,(\psi_n,\psi_{n'}) = e^{-\phi(\lambda_{n'})t}\psi_{n'}(x),\end{aligned} \tag{6.5}$$

where we have used $dK' = e^{-\rho t}dK$ and $(\psi_n,\psi_{n'}) = \delta_{n,n'}$.

Remark 6.1. It is worth noting that, although one can theoretically compute $\partial^2_{KK}C(t,x;K)$ with call prices available at all strikes $K \in (0,\infty)$, a more convenient expression for the integral in (6.12) can be obtained by integrating by parts twice [4] Using the following limits

$$\lim_{K\to\infty} \partial_K C(t,x;K) = 0, \qquad \lim_{K\to\infty} C(t,x;K) = 0, \tag{6.6}$$

$$\lim_{K\to 0} \partial_K C(t,x;K) = -1, \qquad \lim_{K\to 0} C(t,x;K) = x, \tag{6.7}$$

we find

$$\int_0^{\infty} \partial^2_{KK} C(t,x;K)\psi_n(K')dK = \psi_n(0) + x\, e^{-\rho t}\partial_x\psi_n(0) + \int_0^{\infty} C(t,x;K)\partial^2_{KK}\psi_n(K')dK. \tag{6.8}$$

Note that ∂_x is a derivative with respect to the argument of ψ_n whereas ∂^2_{KK} is a derivative with respect to K. The advantage of using expression (6.8) rather than the integral in (6.12) is that the differential operators in (6.8) act on the eigenfunction ψ_n rather than the call price $C(t,x;K)$. Derivatives of ψ_n can be computed analytically, whereas derivatives of call prices $C(t,x;K)$ must be computed numerically from market data.

Now, observe from (4.2) that the value $V^f(t,x)$ of any derivative contract with payoff function $f \in L^2(E,m)$ can be represented as

$$\begin{aligned}V^f(t,x) &= \sum_{n=1}^{\infty}\left(e^{-\phi(\lambda_n)t}\ \psi_n(x)\right)\ c_n^f \\ &= \sum_{n=1}^{\infty}\Big(\int_0^{\infty}\partial^2_{KK}C(t,x;K)\psi_n(K')dK\Big)\ c_n^f, \qquad c_n^f = (\psi_n, f)\,,\end{aligned} \tag{6.9}$$

where, in the second line, we have used (6.5). For example, the sequence of coefficients $c_n = (\psi_n, (\cdot - K)^+)$ allows us to calculate the value of a European call with strike K. Here, two observations are important. First, since the values of $C(t,x;K)$ are obtained from market data, ψ_n and c_n are the only quantities in (6.9) that depend on the parameters of X. Second, $\phi(\lambda_n)$ does not appear in (6.9).

Denote by Φ a parametrization of the background process X. For example, if X is the JDCEV process described in Section 5, then $\Phi = \{a,b,c,\mu,\beta\}$. Let $\psi_n(x;\Phi)$ and $\lambda_n(\Phi)$, respectively, be the nth eigenfunction

[4] Our thanks to Marco Avellaneda for pointing this out.

and eigenvalue of $\mathcal{A}^1 \equiv \mathcal{A}^1(\Phi)$. We can use (6.9) to calibrate the parameters of the Feller process X to a series of call options by numerically solving the following least-squares optimization problem

$$\Phi^* = \arg\min_{\Phi} \sum_{K} \left(\text{IV}[C(t,x,K)] - \text{IV}[\Omega(t,x,K;\Phi)]\right)^2 \tag{6.10}$$

$$\Omega(t,x,K;\Phi) := \sum_{n=1}^{\infty} \Big(\int_0^{\infty} \partial^2_{\kappa\kappa} C(t,x,\kappa)\psi_n(\kappa';\Phi)d\kappa \Big) c_n(K;\Phi), \tag{6.11}$$

$$c_n(K;\Phi) := \left(\psi_n(\cdot;\Phi), (\cdot - K)^+\right),$$

where $\text{IV}[C(t,x,K)]$, indicates the *Black-Scholes implied volatility* corresponding to call price $C(t,x,K)$ and likewise for $\text{IV}[\Omega(t,x,K;\Phi)]$. Note that we have explicitly indicated the dependence of $\Omega(t,x,K;\Phi)$, $c_n(K;\Phi)$ and $\psi_n(\cdot;\Phi)$ on the parameters Φ of the background process X. Observe that for a given set of parameters Φ the value of $\Omega(t,x,K;\Phi)$ can be computed numerically by using (6.8) and by approximating call options $C(t,x;K)$ for every $K \geq 0$ using, e.g., splines of market prices and the limits (6.6)-(6.7). Lastly, we observe that this integral has the term $\kappa' = e^{-\rho t}\kappa = e^{-\phi(-\mu)t}\kappa$ and the value of $\phi(-\mu)$ is not known a priori. In order to circumvent this issue, we define $\theta = e^{-\phi(-\mu)} \in (0,1]$ and introduce this parameter as part of the calibration procedure.

Step 2: Obtaining γ and a non-parametric representation of ν

Assume now that we have performed Step 1 and obtained Φ^*, the optimal parametrization for X in the sense of (6.10). Assume further that further that Φ^* is the *true* parameterization of X so that $\lambda_n = \lambda_n(\Phi^*)$ and $\psi_n(x) = \psi_n(x;\Phi^*)$. The veracity of this assumption will be tested numerically in Section 7.

Solving (6.5) for $\phi(\lambda_n)$, we obtain

$$\phi(\lambda_n) = \frac{-1}{t} \log\left(\frac{\int \partial^2_{KK} C(t,x;K)\psi_n(K')dK}{\psi_n(x)} \right). \tag{6.12}$$

Expression (6.12) gives a non-parametric representation of ϕ in terms of Call prices $C(t,x;K)$ on the market. Thus, we refer to T with Laplace exponent (6.12) as the *market implied Lévy subordinator*.

A parametric representation for X, coupled with a non-parametric representation of ϕ, is sufficient to price European-style options using (4.2). However, we see from (4.3) that, to compute K_{var}, we need γ and ν. We now show how to obtain these objects non-parametrically from ϕ. We must consider two cases (i) compound Poisson subordinators and (ii) infinite activity subordinators.

Compound Poisson subordinators. When the subordinator T is of the compound Poisson type, its Lévy measure $\nu((0,\infty))$ is finite, and it can be written as the product of the net jump intensity $\alpha := \nu((0,\infty))$ times the jump distribution F

$$\nu(ds) = \alpha\, F(ds). \tag{6.13}$$

Using (6.13), the Lévy-Kintchine formula (2.5) can be written

$$\phi(\lambda) = \gamma\,\lambda + \alpha \int_0^{\infty} (1 - e^{-\lambda s}) F(ds) = \gamma\,\lambda + \alpha\left(1 - \widehat{F}(\lambda)\right), \tag{6.14}$$

where we have defined $\widehat{F}(\lambda)$, the Laplace transform of the measure $F(ds)$

$$\widehat{F}(\lambda) := \int_0^{\infty} e^{-\lambda s} F(ds).$$

The drift of the subordinator γ and the net jump intensity α can now be obtained from $\phi(\lambda)$ by taking the following limits

$$\lim_{\lambda\to\infty} \frac{\phi(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \left(\gamma + \alpha \int_0^{\infty} \frac{1 - e^{-\lambda s}}{\lambda} F(ds) \right) = \gamma, \tag{6.15}$$

$$\lim_{\lambda\to\infty} \left(\phi(\lambda) - \gamma\,\lambda\right) = \lim_{\lambda\to\infty} \alpha \int_0^{\infty} \left(1 - e^{-\lambda s}\right) F(ds) = \alpha, \tag{6.16}$$

where we have used $\int_0^\infty F(ds) = 1$. After obtaining γ and α, we use (6.14) to solve for $\widehat{F}$

$$\widehat{F}(\lambda) = 1 + \frac{\gamma\lambda - \phi(\lambda)}{\alpha}. \tag{6.17}$$

Given $\widehat{F}(\lambda)$, one can obtain $F((0,s))$ using the inverse Laplace transform (Bromwich integral)

$$F((0,s)) = \frac{1}{2\pi i}\int_0^s \left(\int_{C-i\infty}^{C+i\infty} e^{\lambda u}\widehat{F}(\lambda)d\lambda\right) du. \tag{6.18}$$

Though we have the value of ϕ, and therefore $\widehat{F}$, only at the eigenvalues λ_n of $\mathcal{A}^1$, there are numerous algorithms for evaluating (6.18) from discrete data; see, e.g., Abate and Whitt (2006).

Infinite activity subordinators. When the subordinator T is *not* of the compound Poisson type, its drift γ can still be found using

$$\lim_{\lambda\to\infty}\frac{\phi(\lambda)}{\lambda} = \lim_{\lambda\to\infty}\left(\gamma + \int_0^\infty \frac{1-e^{-\lambda s}}{\lambda}\nu(ds)\right) = \gamma.$$

To obtain ν we must introduce $\omega(s)$ the *tail of the Lévy measure*

$$\omega(s) := \nu((s,\infty)) = \int_{(s,\infty)}\nu(dz).$$

Following Bertoin (2004), pp. 7, we note that

$$\frac{\phi(\lambda)}{\lambda} - \gamma = \int_{(0,\infty)}\frac{1-e^{-\lambda s}}{\lambda}\nu(ds) = \int_{(0,\infty)} e^{-\lambda s}\omega(s)ds =: \widehat{\omega}(\lambda),$$

Given, $\widehat{\omega}(\lambda)$, one can obtain $\omega(s)$ from the Bromwich integral

$$\omega(s) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{\lambda s}\widehat{\omega}(\lambda)d\lambda.$$

Finally, one obtains $\nu(ds)$ from $\nu(ds) = -d\omega(s)$.

7 Numerical analysis

In this section, we perform a number of numerical studies using the subordinate JDCEV specification in order to verify the applicability of our methods. In particular, we are interested in answering the following questions:

Section 7.1. If we know the true parameters Φ_0 of the JDCEV process X, and if we fix the (finite activity) subordinator T, whose Lévy-Kintchine exponent ϕ_0 is known. Can the calibration procedure described in section 6 recover Φ_0 and ϕ_0 given a series of European call option prices $C(t,x,K)$ generated by these true parameters?

Section 7.2. How sensitive are the drift γ and Lévy measure ν of the subordinator T to misspecifications on the parameters Φ_0 of the background process X? How do these misspecifications affect the value of K_{var}? To answer these questions we generate a set of perturbations around the true parameters Φ_0 of the JDCEV process, then we obtain the market implied subordinator and compute K_{var} for these values.

Section 7.3. How does the calibration method perform with market data? We test our procedure on call prices written on Apple, Google, Microsoft and Facebook.

Section 7.4. How does the spectral method for computing K_{var} compare to alternative numerical methods such as Monte Carlo and finite difference PIDE solvers?

7.1 Calibration to simulated data

In this section, we test how our calibration procedure works on simulated data. We proceed according to the following steps:

Step 1: We fix the background process X as a JDCEV process with parameters $\Phi_0 = \{a_0, b_0, c_0, \mu_0, \beta_0\}$ given by

$$\Phi_0 : \qquad a_0 = 0.2, \qquad b_0 = 0.01, \qquad c_0 = 0.1, \qquad \mu_0 = -0.1, \qquad \beta_0 = -0.1. \tag{7.1}$$

We also fix a finite activity subordinator T with drift $\gamma_0 > 0$, jump arrival rate $\alpha_0 > 0$, and exponentially distributed jumps with mean $1/\eta_0 > 0$.

$$\nu_0(ds) = \alpha_0\eta_0 e^{-\eta_0 s}ds, \qquad \phi_0(\lambda) = \gamma_0\lambda + \frac{\alpha_0\lambda}{\eta_0+\lambda}, \qquad \gamma_0 = 0.5, \qquad \alpha_0 = 2.0, \qquad \eta_0 = 2.0. \tag{7.2}$$

The Laplace exponent of T can be computed in closed form: $\phi_0(\lambda) = \gamma_0\lambda + \left(\frac{\alpha_0\lambda}{\lambda_0+\eta_0}\right)$. Finally, fix an initial stock price $x = 1$ and a time to maturity $t = 1/12$ (one month).

Step 2: Using the spectral representation (5.3) we generate call prices $C(t, x, K)$ with strike prices ranging from $K_{\min}$ to $K_{\max}$ in equally spaced-intervals

$$K_i = K_{\min} + i\Delta K, \qquad \Delta K = (K_{\max} - K_{\min})/N, \qquad i = 0, 1, \cdots, N.$$

Since call prices (5.3) are given as an infinite sum, we fix $\lambda_{\max} > 0$ and sum over the set of eigenvalues that satisfy $|\lambda_n| < \lambda_{\max}$. We emphasize that the call prices are computed using Φ_0 and ϕ_0. The generated prices are taken to be our *market data*.

Step 3: We solve optimization problem (6.10) in order to obtain Φ^*, the best estimate of the true parameters Φ_0 of the background diffusion. The minimization is computed using Mathematica's `FindFit` function. Note that, to solve optimization problem (6.10) we must compute the integral inside (6.11), which can alternatively be written as (6.8). Observe that $\psi_n(0) = 0$ and $\partial_x\psi_n(0)$ are found in (5.2). We use (5.1) to approximate the integral on the right-hand-side of (6.8) using a Riemann sum as follows

$$\int_0^\infty C(t, x; \kappa)\partial^2_{\kappa\kappa}\psi_n(\kappa'; \Phi)d\kappa \approx \sum_{i=1}^{N-1} g_n(K_i; \Phi)\Delta K,$$

where the function g_n is given by

$$g_n(K_i; \Phi) := \theta^{2t}\ C(t, x; K_i)\ \partial^2_{xx}\psi_n(\theta^t K_i; \Phi), \qquad \theta = e^{-\phi(-\mu)}. \tag{7.3}$$

and $\partial^2_{xx}\psi_n(\theta^t K_i; \Phi)$ is given by (5.1). We remark that θ, like the parameters in Φ, is unknown and must be found as part of the optimization procedure.

Step 4: After obtaining a best estimate Φ^* for the parameters of the background process X, we seek a nonparametric approximation ϕ^* for the true Laplace exponent ϕ_0. Specifically, for every $|\lambda_n(\Phi^*)| < \lambda_{\max}$ (see Step 2), we use (6.12) to compute

$$\phi^*(\lambda_n(\Phi^*)) = \frac{-1}{t}\log\left(\frac{\int \partial^2_{KK}C(t, x; K)\psi_n(K'; \Phi^*)dK}{\psi_n(x; \Phi^*)}\right), \tag{7.4}$$

where the integral in (7.4) is obtained using (7.3). Note that we use the parameters Φ^* to compute the eigenfunctions $\psi_n(x; \Phi^*)$ and eigenvalues $\lambda_n(\Phi^*)$.

Step 5: Since the subordinator T is of finite activity, the Lévy measure ν_0 factors into a product of the jump intensity α_0 and the jump distribution F_0. Thus, we use equations (6.15), (6.16) and (6.17) to obtain an approximation for the drift γ_0, jump intensity α_0 and Laplace transform of the jump distribution $\widehat{F}_0$. We denote these approximations by γ^*, α^* and $\widehat{F}^*$, respectively. Specifically, we take

$$\begin{aligned}
\gamma^* &:= \frac{\phi^*(\lambda_{\max}(\Phi^*))}{\lambda_{\max}(\Phi^*)}, \\
\alpha^* &:= \phi^*(\lambda_{\max}(\Phi^*)) - \gamma^*\lambda_{\max}(\Phi^*) \\
\widehat{F}^*(\lambda_n(\Phi^*)) &:= 1 + \frac{\gamma^*\lambda_n(\Phi^*) - \phi^*(\lambda_n(\Phi^*))}{\alpha^*},
\end{aligned}$$

where $\lambda_{\max}(\Phi^*)$ is the largest $\lambda_n(\Phi^*)$ such that $|\lambda_n(\Phi^*)| < \lambda_{\max}$.

Step 6. Use the results of propositions 5.2 and 5.3 to calculate K^*_{var}, the estimate of K_{var}.

Results. After performing Steps 1-3 described above, we obtain the following parameters of the background JDCEV diffusion

$$\Phi^* : \qquad a^* = 0.2021, \qquad b^* = 0.0112, \qquad c^* = 0.1953, \qquad \mu^* = -0.1004, \qquad \beta^* = -0.1023. \tag{7.5}$$

Using the parameters Φ^*, we then compute the approximation ϕ^* for the Lévy-Kintchine exponent (step 4), which can be used to the approximate drift γ^*, jump intensity α^* and Laplace transform of the jump distribution $\widehat{F}^*$ (step 5). We obtain

$$\gamma^* = 0.5151, \qquad \alpha^* = 1.9402. \tag{7.6}$$

In Figure 1 we plot the following

$$\phi_0(\lambda) \qquad \text{vs} \qquad \phi^*(\lambda_n(\Phi^*)), \tag{7.7}$$

$$\widehat{F}_0(\lambda) = 1 + \frac{\gamma_0\lambda - \phi(\lambda)}{\alpha} \qquad \text{vs} \qquad \widehat{F}^*(\lambda) = 1 + \frac{\gamma^*\lambda_n(\Phi^*) - \phi^*(\lambda_n(\Phi^*))}{\alpha^*}, \tag{7.8}$$

Lastly, we compute the true and approximate values of K_{var}

$$K_{\text{var}} = 0.0053, \qquad K^*_{\text{var}} = 0.0050. \tag{7.9}$$

Comparing (7.1) with (7.5), we see that the estimated values of μ^*, a^* and β^* fall within 1% of their true values. The estimate b^* differs approximately 10% from the true value. The only estimated parameter of the JDCEV process X that differs significantly from its true value is c^*. The estimates for the parameters of the subordinator T are similarly convincing. Comparing (7.2) and (7.6), we see that the estimates γ^* and α^* deviate roughly 3% from their true values. Moreover, Figure 1 shows a close match between ϕ and ϕ^*. Finally, and most importantly, the estimated VS rate K^*_{var} falls within 6% of its true value.

7.2 Sensitivity analysis

In the calibration experiment discussed above, we obtained an estimate Φ^* of the diffusion parameters Φ_0 as well as a nonparametric estimate ϕ^* of the Laplace exponent ϕ_0 of the subordinator T. This enabled us to compute an estimate K^*_{var} of the true VS rate K_{var}. We now ask the question: how sensitive are ϕ^* the value of K^*_{var} to misspecification of these model parameters? To answer this question we proceed according to the following steps:

Step 1. Generate a series of call prices under the true parameters Φ_0 and ϕ_0 (i.e., steps 1 and 2 of section 7.1). Again, we assume that the generated prices given by the market. As such, these are the prices to which we will calibrate our parameters.

Step 2. Choose one of the true parameters $\chi_0 \in \Phi_0 = \{a_0, b_0, c_0, \mu_0, \beta_0\}$ and generate a series of perturbed parameter sets $\widetilde{\Phi}(\chi_\varepsilon)$ by replacing χ_0 with $\chi_\varepsilon = \chi_0(1+\varepsilon)$ while holding all other parameters constant. For example $\widetilde{\Phi}(b_\varepsilon) = \{a_0, b_0(1+\varepsilon), c_0, \mu_0, \beta_0\}$ and like wise for other perturbed parameter sets.

Step 3. Assume that $\widetilde{\Phi}(\chi_\varepsilon)$ corresponds to the best estimate of Φ_0. That is, we assume $\widetilde{\Phi}(\chi_i) = \Phi^*$ in step 3 of section 7.1 Under this assumption, obtain ϕ^*, γ^*, α^* and $\widehat{F}^*$ using steps 4 and 5 of section 7.1.

Step 4. Use the results of propositions 5.2 and 5.3 to calculate K^*_{var} under these estimations.

Results. The previous steps return the values of γ^*, α^* and K^*_{var} and nonparametric maps of ϕ^* and $\widehat{F}^*$assuming that the parameter χ has been misspecified. The values of obtained for γ^*, α^* and K^*_{var} in our numerical experiments are given in Table 1. Plots of ϕ^* and $\widehat{F}^*$ are given in Figure 2. From Table 1 it is clear that misspecification of a_0 causes α^*, γ^*, and K^*_{var} to deviate most drastically from their true values. Table 1 also shows that decreasing a_0 has the effect of increasing γ^* and α^*. In fact, this is not unexpected. From (2.8) and $\sigma(x) = a_0 x^\beta$ we see that coefficient of the diffusion (i.e., second derivative) term in $\mathcal{A}^{\phi,\beta}$ is proportional to γa_0^2. Likewise, a smaller a_0 results in a more narrowly peaked density p^β,

χ_0		$\varepsilon = -0.10$	$\varepsilon = -0.05$	$\varepsilon = 0.00$	$\varepsilon = 0.05$	$\varepsilon = 0.10$
a_0	α^*/α_0	1.581	1.264	1.000	0.782	0.605
	γ^*/γ_0	1.017	1.008	1.000	0.990	0.977
	$K^*_{\text{var}}/K_{\text{var}}$	0.959	0.978	1.000	1.026	1.058
b_0	α^*/α_0	0.984	0.992	1.000	1.008	1.016
	γ^*/γ_0	1.004	1.002	1.000	0.998	0.996
	$K^*_{\text{var}}/K_{\text{var}}$	1.014	1.007	1.000	0.993	0.986
μ_0	α^*/α_0	0.775	0.890	1.000	1.112	1.232
	γ^*/γ_0	1.097	1.046	1.000	0.959	0.921
	$K^*_{\text{var}}/K_{\text{var}}$	1.013	1.006	1.000	0.996	0.992

Table 1: These above table demonstrates how misspecification of one of the parameters (a_0, b_0, or μ_0) of the process X affects the estimates α^* and γ^* and K^*_{var}. We fix $\chi_0 \in \{a_0, b_0, \mu_0\}$ and perturb it according to $\chi_0 \to \chi_\varepsilon = (1+\varepsilon)\chi_0$. We then obtain α^* and γ^* and K^*_{var} according to steps 4, 5 and 6 of section 7.1. We do not include the results for β_0 and c_0 since these parameters had less of an effect on α^*, γ^* and K^*_{var} than did a_0, b_0 and μ_0.

and hence, from (2.9) a more narrowly peaked Lévy kernel $\pi^{\phi,\beta}$. This effect can be partly compensated for by increasing the jump-intensity α_0 of the subordinator. Though, these effects are not isolated, and γ, α and a appear in numerous terms in $\mathcal{A}^{\phi,\beta}$. While misspecification of a_0 had the most drastic affect on the estimated parameters of the subordinator T, we note that in Section 7.1 we obtained an estimate a^* within 1% of its true value a_0. Moreover, the values obtained for K^*_{var} using the perturbed parameter sets $\widetilde{\Phi}(\chi_\varepsilon)$, never deviates more than 6% from the true value K_{var}.

7.3 Calibration to market data

We now test our methodology with real data. We use call options prices on Citi, Facebook and Google, to estimate Φ^* and ϕ^* for each firm. We then use this information to compute VS rates. Call option quotes were obtained on January 30, 2015 from Google Finance and have a maturity of 1 month. None of the firms paid dividends over the tenor of the option. For each individual symbol we implement the procedure described in section 7.1, with one difference: we must smooth the noisy call data.

Data Smoothing. Let $C_M(t, K_i)$ denote the price of a call option with maturity t and strike K_i, as quoted on the market. First, for each call price in the data set $C_M(t, K_i)$, we generate a Black-Scholes implied volatility $\text{IV}[C_{raw}(t, K_i)]$. We then use the *Stochastic Volatility Inspired* (SVI) parametric family of Gatheral (2004) and Gatheral and Jacquier (2012) to interpolate the volatility smile. Specifically we seek

$$\Gamma^* = \arg\min_{\Gamma} \sum_i \left(\text{IV}[C_{\text{M}}(t, K_i)] - \sigma_{\text{SVI}}(t, K_i; \Gamma)\right)^2,$$

$$\sigma_{\text{SVI}}(t, x, K; \Gamma) = \left\{ \frac{a}{t} + \frac{b}{t}\left(\rho(\log(K/x) - m) + \sqrt{((\log(K/x) - m)^2 + \xi^2}\right)\right\}^{1/2},$$

$$\Gamma = \{a, b, m, \rho, \xi : a, b \geq 0,\ \xi > 0,\ m \in \mathbb{R},\ \rho \in [-1, 1]\}.$$

A smooth surface of call prices is then obtained by defining

$$C(t, x, K) := C_{\text{BS}}(t, x, K; \sigma_{\text{SVI}}(t, K; \Gamma^*)), \tag{7.10}$$

where $C_{\text{BS}}(t, x, K; \Sigma)$ is the Black-Scholes call price of a call with initial stock price x, maturity t, strike price K and volatility Σ. Note also that, by the definitions of $C(t, x, K)$ and implied volatility, we have $\text{IV}[C(t, x, K)] = \sigma_{\text{SVI}}(t, x, K; \Gamma^*)$. Upon smoothing the raw data we follow steps 2 to 6 from section 7.1 to obtain Φ^*, ϕ^* and K^*_{var}.

Results: In Figure 3 we plot the raw implied volatility data for the three stocks under consideration as well as the SVI interpolation of the data. Market prices are then defined according to (7.10). Next, we perform steps 2 and 3 from Section 7.1 in order to obtain the parameters Φ^* of the JDCEV process X. We obtain

	x	a^*	b^*	c^*	μ^*	β^*
CITI	46.95	0.3088	0.1046	1.3819	-0.1517	-0.1204
FB	75.91	0.2010	0.0071	0.0919	-0.0321	- 0.1125
GOOG	534.52	0.4151	0.0343	1.3102	-0.4498	-0.1523

With parameter estimate Φ^* in hand, we proceed to Steps 4 and 5 from Section 7.1 to obtain non-parametric estimates of ϕ^* and $\widehat{F}^*$. We plot these estimates in Figure 4. Lastly, we use Propositions 5.2 and 5.3 to calculate one-month VS rates. For three stocks under investigation, we obtain

	CITI	FB	GOOG
K^*_{var}	0.00523831	0.0065024	0.0046593

7.4 Comparison to other pricing and calibration methods

In this section we compare our pricing, modeling and semi-parametric calibration methodology to some alternative methods.

Comparison to time-changed Lévy framework

Carr et al. (2012) model the stock price as $S_t = \exp(L_{\tau_t})$ where L is a Lévy process and τ is a continuous increasing random process (i.e., a stochastic time-change). In this setting, they show that the VS rate is given by a European log-style contract $K_{\text{var}} = Q_L \mathbb{E}\left[-\log\left(S_t/S_0\right)\right]$ where Q_L is a constant that depends only on the Lévy triplet of L and not on the time-change τ. Like our Lévy subordinated diffusion framework, the time-changed Lévy framework of Carr et al. (2012) permits S to experience both jumps and stochastic volatility. However, the time-changed Lévy framework does not allow for the possibility of default.

The time-changed Lévy framework is semi-parametric in that, for a fixed Lévy process L the VS rate K_{var} can be computed relative to the value of a European log contract with no knowledge of the time-change process τ. However, from a practical standpoint, there is no way to determine what Lévy process L one should use to model S. The authors do attempt to shed some light on this issue. For a parametrically defined CIR time-change, which is independent of L, the authors estimate Q_L from S&P500 data and find that Q_L lies roughly between 1.9 and 2.2. This limits the choice of Lévy processes somewhat. However, a different specification of the time-change process may have led to a different estimates for Q_L. Since the authors provide no method of determining the time-change process τ, one can question whether the "true" Q_L lies outside of the interval $(1.9, 2.2)$.

By contrast, in the Lévy subordinated diffusion framework promoted here, we provide a method of calibrating the parameters of the background process X and nonparametrically determine drift γ and Lévy measure ν of the subordinator T.

Comparison to Monte Carlo pricing

In a standard Monte Carlo setting, one discretizes the time interval $[0, t]$ into discrete steps: $t_i = i\delta_t$ $(i = 0, 1, 2, \ldots, M)$ with $\delta_t := t/M$. Since, as $M \to \infty$ (i.e., $\delta_t \to 0$), $\sum_{i=0}^{M-1}(1 - D^{\phi}_{t_{i+1}}) \log^2(S_{t_{i+1}}/S_{t_i}) \longrightarrow \int_0^t (1 - D^{\phi}_u) d[\log(S)]_u$ in probability (see, e.g., Barndorff-Nielsen and Shephard (2002)), then $\mathbb{E}\big[\sum_{i=0}^{M-1}(1 - D^{\phi}_{t_{i+1}}) \log^2(S_{t_{i+1}}/S_{t_i})\big] \to \mathbb{E}\big[\int_0^t (1 - D^{\phi}_u) d[\log(S)]_u\big]$. Hence, an estimate $\widehat{K}_{\text{var}}$ of the VS rate can be computed using Monte Carlo as,

$$\widehat{K}_{\text{var}} := \frac{1}{N} \sum_{j=1}^{N} \sum_{i=0}^{M-1} \mathbb{I}_{\{\zeta^{\phi} > t_{i+1}\}} \log^2 \left(\frac{S^{(j)}_{t_{i+1}}}{S^{(j)}_{t_i}} \right), \tag{7.11}$$

where $S^{(j)}$ is the jth sample path. The Monte Carlo method can be useful for a few reasons. First, approximate sample paths can be simulated for almost any semimartingale process. Second, one does not need to approximate the VS payoff by the quadratic variation of $\log S$. Instead, one could choose $\delta_t = 1$ day (for VSs on indices, this is the typical sampling frequency). By contrast, in the Lévy subordinated diffusion framework, we require a killed diffusion $(1-D)X$ whose generator has a closed-form eigenfunction expansion, and we must approximate the discretely monitored VS with the continuously monitored approximation.

However, Monte Carlo methods are not without their drawbacks. First, exact simulation of sample paths is not possible for most semimartingales; simulation of infinite activity semimartingales, in particular, can

be problematic. Second, even when exact simulation is possible, the estimate $\widehat{K}_{\text{var}}$ is random; the best one can hope for is a small confidence interval, which can be accomplished by increasing the sample size N. Unfortunately, the size of the confidence interval is proportional to the standard deviation of the payoff and inversely proportional to $\sqrt{N}$. Thus, decreasing the size of the confidence interval by a factor of z requires multiplying the number of sample paths by z^2.

In order to compare the computational costs and relative accuracy of computing VS rates using Monte Carlo and eigenfunction expansion techniques, we compute K_{var} using these two methods. As in Section 7.1, we fix X as a JDCEV process and T as a finite activity subordinator with exponentially distributed jumps. Specifically, we fix $t = 1/4$ and $S_0 = x = 80$, and use the following parameters:

$$a = 143.108, \quad b = 0.001, \quad c = 0.046, \quad \beta = -1.5, \quad \mu = -0.16, \quad \alpha = 2.8, \quad \gamma = 2.25, \quad \eta = 1.$$

We first compute the VS rate K_{var} using the spectral expansions (5.5) and (5.6). Since there are three different summations in these equations that may converge at a different rate (i.e., these may require a different number of iterations to converge), we need to count our number of iterations in a particular way. Let n_{d} is the number of iterations required in (5.5) such that the relative error, RE,[5] is less or equal to some tolerance level $\epsilon > 0$. Similarly, we define n_{j_1} as the number of iterations required in the first term of (5.6), and n_{j_2} and m_{j_2} as the number of iterations required in each summation of the second term of (5.6). Therefore, the total number of iterations, $\mathcal{N}$, required for all three summations converge given a particular tolerance level $\epsilon > 0$ is $\mathcal{N} = n_{\text{d}} + n_{\text{j}_1} + n_{\text{j}_2} \times m_{\text{j}_2}$. The *"true VS rate"*, $K_{\text{var}}^{\text{true}} = 0.219308$, is computed using the same procedure with $\epsilon = 10^{-11}$. The following table illustrates the results of this experiment including the *absolute percentage error*, $\text{APE}\,(\%) := 100 \times |(K_{\text{var}} - K_{\text{var}}^{\text{true}})/K_{\text{var}}^{\text{true}}|$, which we use as a measure of accuracy.

Spectral Method

ϵ	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}
$\mathcal{N}$	55,805	106,415	239,657	316,899	374,257	457,113
K_{var}	0.300305	0.251197	0.217001	0.219781	0.219748	0.219374
APE (%)	36.93	14.54	1.05	0.22	0.20	0.03

Next, we estimate K_{var} using the Monte Carlo simulation we use the algorithm described in Appendix A. We divide the time interval $[0, t]$ into $M = 100$ equally spaced subintervals. In order to make a fair comparison, the total number of iterations $\mathcal{N}$ is computed as $\mathcal{N} = M \times N$ where, N is the number of sample paths. Below, we report the obtained estimates $\widehat{K}_{\text{var}}$ as well as the corresponding APE (%).

Monte Carlo

$\mathcal{N}$	10^4	10^5	10^6	10^7	10^8
$\widehat{K}_{\text{var}}$	0.176496	0.255398	0.236924	0.230999	0.229
APE (%)	19.52	16.46	8.03	5.33	4.42

The above results are summarized in Figure 5. It is clear the that the Monte Carlo scheme is neither as accurate, nor as computationally efficient as the eigenfunction expansion, which converges at an exponential rate. Note that we cannot expect the VS rate as computed via Monte Carlo $\widehat{K}_{\text{var}}$ to exactly coincide with the VS rate computed by eigenfunction methods K_{var}, as the Monte Carlo computation contains an error due to discrete sampling.

Comparison to finite difference PIDE solvers

Suppose a stock $S = \exp(Y)$ where Y is a general, time-homogenous scalar Markov process

$$dY_t = b(Y_t)dt + a(Y_t)dW_t + \int_{\mathbb{R}} \xi(N^Y(dt, d\xi) - \nu(Y_t, d\xi)dt).$$

Here, for simplicity, we assume that all jumps can be compensated and we ignore to possibility of default. These simplifying assumptions are not central to the analysis that follows below. Let $Z = [Y]$ be the

[5]Relative error is defined as $RE := |a_n/s_n|$, where $s_n = \sum_{i=1}^{n} a_i$ is the partial sum up to the nth term, a_n.

quadratic variation of the Y process. The Lévy-Itô SDE for Z is

$$dZ_t = \left(a^2(Y_t) + \int_{\mathbb{R}} \xi^2 \nu(Y_t, d\xi)\right) dt + \int_{\mathbb{R}} \xi^2 (N^Y(dt, d\xi) - \nu(Y_t, d\xi)dt).$$

Under suitable conditions, the VS rate $K_{\text{var}} = \mathbb{E}_{y,z}[\log S]_t = \mathbb{E}_{y,z} Z_t =: v(t, y, z)$ corresponds to the solution of the following partial integro-differential equation (PIDE)

$$(-\partial_t + \mathcal{A})v = 0, \qquad v(0, y, z) = z, \tag{7.12}$$

where $\mathcal{A}$, the generator the (Y, Z) process, is given by

$$\mathcal{A} = b(y)\partial_y + \frac{1}{2}a^2(y)\partial_{yy}^2 + \left(a^2(y) + \int_{\mathbb{R}} \xi^2 \nu(Y_t, d\xi)\right)\partial_z + \int_{\mathbb{R}} \nu(y, d\xi)\left(\theta_\xi^y \theta_{\xi^2}^z - 1 - \xi\partial_y - \xi^2\partial_z\right).$$

Here, θ_ξ^y is the shift operator in the y-direction and $\theta_{\xi^2}^z$ is the shift operator in the z direction

$$\theta_\xi^y f(y, z) = f(y + \xi, z), \qquad \theta_{\xi^2}^z f(y, z) = f(y, z + \xi^2).$$

If PIDE (7.12) does not have an analytic solution, then one can seek a numerical solution using a finite difference scheme. However, finite difference PIDE solvers may not be as numerically efficient as eigenfunction methods when the latter is possible. To see this, Figure 5 shows that the errors under spectral expansion decrease exponentially with respect to the number of iterations used. Meanwhile, in a finite difference scheme the errors in terms of spatial and time discretization are usually polynomial (see, for instance, Feng and Linetsky (2008) for a detailed analysis). The convergence can be significantly affected by the state-dependence of the coefficients and the behavior at the boundaries. This is particularly problematic for any JDCEV-like process, whose drift and diffusion coefficients tend to infinity as x approaches zero. Boundary behavior is not an issue with eigenfunction methods.

8 Conclusion

In this paper we model the price process of an underlying S as a Feller diffusion time changed by a Lévy subordinator. This class of models, first developed in Mendoza-Arriaga et al. (2010), allows for the underlying to experience jumps with a state-dependent Lévy measure, local stochastic volatility and a local stochastic default intensity. In this setting, we make four significant contributions: (i) we propose a modified VS contract that is valid for any asset whose value can jump to zero, (ii) we derive an expression for the VS rate, (iii) we give the canonical representation of S, which is needed for VS computations, and (iv) provide a semiparametric calibration procedure. Finally, we provide extensive numerical tests in order to verify that our calibration procedure can be carried out in practice.

Thanks

The authors would like to thank Stephan Sturm, Peter Carr, Liming Feng and Vadim Linetsky for their helpful comments on this work. The authors would also like to thank an anonymous referee and an anonymous associate editor for detailed suggestions, which greatly improved the quality of this manuscript.

A Simulation of the subordinate JDCEV process

In this section we provide an algorithm for exact simulation of the JDCEV process. First, we observe that Theorem 2.3.1 of Mendoza-Arriaga (2009) allows to transform the expectations of the JDCEV process with killing to expectation of a CEV process, and hence, removing the killing term from the expectation. For the reader's convenience, we reproduce this theorem here.

Theorem A.1. *Suppose X follows the JDCEV process of section 5 with $\mu \in \mathbb{R}$, $\beta < 0$, $a > 0$, and $b, c \geq 0$, $X_0 = x > 0$, prior to the first hitting time $T_L^X > t$ of level $L \geq 0$. Then, for any function f for which the expectations exists, we have*

$$\mathbb{E}_x\left[\mathbb{I}_{\{T_L^X > t\}} e^{-\int_0^t k(X_u)du} f(X_t)\right] = z^{-\frac{2c}{2c+1}} e^{-\xi t} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{T_{L^{2c+1}}^Z > t\}} Z_t^{\frac{2c}{2c+1}} f\left(Z_t^{\frac{1}{2c+1}}\right)\right],$$
$$\text{with} \quad Z_0 = z = x^{2c+1} > 0, \quad \xi := 2c(\mu + b) + b; \tag{A.1}$$

where $T^Z_{L^{2c+1}}$ is the first hitting time of Z to level $L^{2c+1} \geq 0$, and $\widetilde{\mathbb{E}}$ is the expectation under the law of the standard CEV process $Z = \{Z_t, t \geq 0\}$ solving the SDE

$$dZ_t = \widetilde{\mu} Z_t dt + \widetilde{a} Z_t^{\widetilde{\beta}+1} d\widetilde{B}_t, \quad Z_0 = z = x^{2c+1} > 0, \tag{A.2}$$

with parameters

$$\widetilde{\mu} := (2c+1)(\mu + b), \qquad \widetilde{\beta} := \frac{\beta}{2c+1}, \qquad \widetilde{a} := (2c+1)a.$$

Proof. Let $Z_t = X_t^{2c+1}$ and apply Ito's formula to obtain

$$dZ_t = [\widetilde{\mu} + 2c(2c+1)a^2 Z_t^{2\widetilde{\beta}}] Z_t dt + \widetilde{a} Z_t^{\widetilde{\beta}+1} dB_t = \widetilde{\mu} Z_t + \widetilde{a} Z_t^{\widetilde{\beta}+1} [dB_t + 2ca Z_t^{\widetilde{\beta}} dt].$$

This suggest the change of probability measure such that $\widetilde{B}_t = B_t + \int_0^t 2ca Z_u^{\widetilde{\beta}} du$ is a standard Brownian motion under the new measure, so that the process Z is a standard CEV. Then,

$$\mathbb{E}_x \Big[\mathbb{I}_{\{T_L^X > t\}} e^{-\int_0^t k(X_u) du} f(X_t) \Big] = \mathbb{E}_z \Big[\mathbb{I}_{\{T^Z_{L^{2c+1}} > t\}} e^{-bt - ca^2 \int_0^t Z_u^{2\widetilde{\beta}} du} f\big(Z_t^{\frac{1}{2c+1}}\big) \Big] \tag{A.3}$$

$$= \widetilde{\mathbb{E}}_z \Big[\mathbb{I}_{\{T^Z_{L^{2c+1}} > t\}} e^{-bt - ca^2 \int_0^t Z_u^{2\widetilde{\beta}} du + 2ca \int_0^t Z_u^{\widetilde{\beta}} d\widetilde{B}_u - \frac{(2ca)^2}{2} \int_0^t Z_u^{2\widetilde{\beta}} du} f\big(Z_t^{\frac{1}{2c+1}}\big) \Big] \tag{A.4}$$

In (A.3) we use the fact that $\inf\{t : X_t = L\} = \inf\{t : Z_t = L^{2c+1}\}$. In (A.4) we used Girsanov's theorem to change the probability to the new measure under which $\widetilde{B}$ is a standard Brownian motion and Z follows the standard CEV ($\widetilde{\mathbb{E}}$ is the expectation under this new measure, and $e^{2ca \int_0^t Z_u^{\widetilde{\beta}} d\widetilde{B}_u - \frac{(2ca)^2}{2} \int_0^t Z_u^{2\widetilde{\beta}} du}$ is the corresponding Radon-Nikodym density). From (A.2) we can write

$$Z_t = z e^{\widetilde{\mu} t + \int_0^t \widetilde{a} Z_u^{\widetilde{\beta}} d\widetilde{B}_u - \frac{\widetilde{a}^2}{2} \int_0^t Z_u^{2\widetilde{\beta}} du}.$$

Therefore,

$$Z_t^{\frac{2c}{2c+1}} z^{-\frac{2c}{2c+1}} e^{-2c(\mu+b)} = e^{2ca \int_0^t Z_u^{\widetilde{\beta}} d\widetilde{B}_u - c(2c+1)a^2 \int_0^t Z_u^{2\widetilde{\beta}} du}. \tag{A.5}$$

Lastly, substituting (A.5) into (A.4) we obtain (A.1). □

Recall that for $-1/2 \leq \widetilde{\beta} < 0$, zero is an exit boundary for CEV process Z (i.e., zero is an absorbing state), while for $\widetilde{\beta} \leq -1/2$, zero a regular boundary, and thus, a killing (absorbing) condition at $Z = 0$ needs to be imposed. Therefore, let $L = 0$, then $\mathbb{I}_{\{T_0^Z > t\}} = \mathbb{I}_{\{Z_t > 0\}}$. Hence, applying Theorem A.1 we have,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{M-1} \mathbb{I}_{\{\zeta^\phi > t_{i+1}\}} \mathbb{I}_{\{T_0^X > T_{t_{i+1}}\}} \log^2\left(\frac{S_{t_{i+1}}}{S_{t_i}} \right) \right] \\
&\quad = \sum_{i=0}^{M-1} \mathbb{E}\left[\mathbb{I}_{\{\zeta^\phi > t_{i+1}\}} \mathbb{I}_{\{T_0^X > T_{t_{i+1}}\}} \log^2\left(\frac{S_{t_{i+1}}}{S_{t_i}} \right) \right] \\
&\quad = \sum_{i=0}^{M-1} \mathbb{E}\left[\mathbb{I}_{\{\zeta^\phi > t_{i+1}\}} \mathbb{I}_{\{T_0^X > T_{t_{i+1}}\}} \log^2\left(\frac{e^{\rho t_{i+1}} X_{T_{t_{i+1}}}}{e^{\rho t_i} X_{T_{t_i}}} \right) \right] \\
&\quad = \frac{z^{-\frac{2c}{2c+1}}}{(2c+1)^2} \sum_{i=0}^{M-1} \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{T_0^Z > T_{t_{i+1}}\}} e^{-\xi T_{t_{i+1}}} Z_{T_{t_{i+1}}}^{\frac{2c}{2c+1}} \log^2\left(\frac{e^{(2c+1)\rho t_{i+1}} Z_{T_{t_{i+1}}}}{e^{(2c+1)\rho t_i} Z_{T_{t_i}}} \right) \right],
\end{aligned}$$

Therefore, we can rewrite (7.11) as,

$$\widehat{K}_{\text{var}} = \frac{z^{-\frac{2c}{2c+1}}}{(2c+1)^2} \frac{1}{N} \sum_{j=1}^{N} \sum_{i=0}^{M-1} \mathbb{I}_{\left\{ Z^{(j)}_{T^{(j)}_{t_{i+1}}} > 0 \right\}} e^{-\xi T^{(j)}_{t_{i+1}}} \left(Z^{(j)}_{T^{(j)}_{t_{i+1}}} \right)^{\frac{2c}{2c+1}} \log^2\left(\frac{e^{(2c+1)\rho t_{i+1}} Z^{(j)}_{T^{(j)}_{t_{i+1}}}}{e^{(2c+1)\rho t_i} Z^{(j)}_{T^{(j)}_{t_i}}} \right),$$

with $z = x^{2c+1} > 0$, $\rho = \phi(-\mu)$, and where $(Z^{(j)}, T^{(j)})$ indicates the jth sample path of (Z, T). The sample paths of (Z, T) can be simulated over the time grid $0 = t_0 < t_1 < \cdots < t_M = t$ as follows:

Step 1. Generate the sample path of the subordinator as $T_{t_1} \leq T_{t_2} \leq \cdots \leq T_{t_M}$ (recall that $T_0 = 0$) using standard algorithms (see, e.g., chapter 6 of Cont and Tankov (2004))
Step 2. For each $i = 1, 2, \ldots, M$, generate the sample path of $Z = \{Z(T_{t_1}), Z(T_{t_2}), \ldots, Z(T_{t_M})\}$. Sample paths of Z, with absorption at zero, can be simulated as proposed in, e.g., Makarov and Glew (2010) and Lindsay and Brecher (2012).

We observe that when the subordinator T is of finite activity, then the previous steps provide us with an exact simulation algorithm for obtaining the VS rate estimate $\widehat{K}_{\text{var}}$.

B Proofs

B.1 Proof of Lemma 4.1

From Eq. (2.7) we observe that $\mathbb{E}[(1 - D_t^\phi) f(X_t^\phi)] = (1-d)\mathcal{P}_t^{\phi,1} f(x)$. Therefore, the first term of Eq. (4.3) follows immediately. Using the definition (2.9) of $\pi^\phi(x, y)$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \log^2\Big(1 + \frac{y}{X_{u-}^\phi}\Big)\pi^{\phi,1}(X_{u-}^\phi, y)dy \\
&= \int_{(0,\infty)} \Big(\int_{E\setminus\{x\}} \log^2\Big(\frac{y}{X_{u-}^\phi}\Big) p^1(s, X_{u-}^\phi, y)dy\Big)\nu(ds) \\
&= \int_{(0,\infty)} \big(\mathcal{P}_s^1 \log^2(X_{u-}^\phi) - 2\log(X_{u-}^\phi)\mathcal{P}_s^1 \log(X_{u-}^\phi) + \log^2(X_{u-}^\phi)\mathcal{P}_s^1 1\big)\nu(ds). \qquad \text{(B.1)}
\end{aligned}$$

We justify the exchange of integrals in (B.1) as follows. From Proposition 32.5(iii) in Sato (1999), p.215, we know that if $\|\mathcal{P}_t f(x) - f(x)\| = O(t)$ as $t \downarrow 0$, then $\int_{\mathbb{R}} f(y)\pi^\phi(x, y)dy = \int_{(0,\infty)}(\mathcal{P}_s f(x) - f(x))\nu(ds)$. Observe that $f(y) = \log^2(y/x) = O(|x - y|^2)$ as $y \to x$, then by continuity of X_t (i.e., $X_t \to x$ as $t \to 0$), to prove $\|\mathcal{P}_t^1 f(x) - f(x)\| = O(t)$ it suffices to show that $\int_E (y - x)^2 p^1(t, x, y)dy = O(t)$ as $t \downarrow 0$. Indeed, the latter holds true since for an arbitrary $\epsilon > 0$, we have $\int_E \mathbb{I}_{\{|x-y|<\epsilon\}}(y - x)^2 p^1(t, x, y)dy \leq Ct$ as $t \downarrow 0$ (cf., McKean (1956), Theorem 4.5). Therefore, since $\int_{(0,\infty)}(s \wedge 1)\nu(ds) < \infty$, we can apply Fubini's theorem. The claim now follows from observing that $\mathbb{E}[(1 - D_u^\phi) f(s, X_u^\phi)] = (1-d)\mathcal{P}_u^{\phi,1} f(s, x)$.

B.2 Computation of $\tilde{c}_n = (x^p, \psi_n)$ from Proposition 5.1

Part (ii). Consider the series $U(x) = \sum_{k=1}^\infty u_k(x)$ for $x \in D \subset (0, \infty)$. If for all $x \in D$ the function $u_k(x)$ satisfies the inequality $|u_k(x)| \leq d_k$ for $k = 1, 2, \cdots$, where the series $\sum_{k=1}^\infty d_k < \infty$, then the series $U(x)$ converges uniformly (see Prudnikov et al. (1990), Section I.3.4.3, p.751). From the inequality (27a) on p.54 of Nikiforov and Uvarov (1988), we find that $|\psi_n^1(x)| < C/(n-1)^{1/4}$ for some $C < \infty$ independent of n. Therefore, we have

$$|c_n \psi_n(x)| \leq \frac{C\Big|\left(\frac{1-p}{2|\beta|}\right)_{n-1}\Big|}{\sqrt{(n-1)!\Gamma(\nu+n)}(n-1)^{1/4}} = d_n,$$

To show that the series $\sum_n^\infty d_n$ converges, it is enough to show that for a large n we have $\log(d_n)/\log(n) < -1$ (see Prudnikov et al. (1990) Section I.3.2.19, p.751). Therefore, observe that for large values of n, i.e., $n \gg 1$, we have $d_n \approx \frac{(n-1)^{\frac{1-p}{2|\beta|}+n/2-3/2}}{(\nu+n)^{(\nu+n)/2-1/4}}$. Thus

$$\begin{aligned}
\frac{\log(d_n)}{\log(n)} &\approx \frac{\Big(\left(\frac{1-p}{2|\beta|}\right) + n/2 - 3/2\Big)\log(n-1) - ((\nu+n)/2 - 1/4)\log(\nu+n)}{\log(n)} \\
&< \Big[\left(\left(\frac{1-p}{2|\beta|}\right) + \frac{n}{2} - \frac{3}{2}\right) - \left(\frac{\nu+n}{2} - \frac{1}{4}\right)\Big]\frac{\log(n-1)}{\log(n)}.
\end{aligned}$$

Note that $\log(n-1)/\log(n) \uparrow 1$. Moreover, it can be verified that the term inside the bracket is less than -1 for all $p > (\beta+1)/2 - c$ (recall $\nu = (2c+1)/(2|\beta|)$). This shows uniform convergence at $t = 0$ for $p > (\beta+1)/2 - c$. To show that the series is absolutely convergent at $t = 0$ it suffices to show that $\lim_{n\to\infty} d_{n+1}/d_n < 1$ with $d_n = |c_n \psi_n(x)|$ (i.e., d'Alambert's test for convergence). Equivalently, $\sum_n^\infty d_n$ converges if $\log(d_{n+1}/d_n) < 0$ as $n \to \infty$. Hence, analyzing the asymptotic behavior and noticing that $L_{n-1}^\nu(z) \approx e^{\frac{z}{2}} z^{-(2\nu+1)/4}(n-1)^{\nu/2-1/4}\cos\{2\sqrt{(n-1)z} - \pi(2\nu+1)/4\}/\sqrt{\pi}$ for $n \gg 1$, we find that

$$d_n = \Big| \frac{A^{\frac{\nu}{2} - \frac{p+2c}{2|\beta|}} \left(\frac{1-p}{2|\beta|}\right)_{n-1} \Gamma\left(\frac{p+2c}{2|\beta|}+1\right)}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}} \psi_n(x) \Big| < C \frac{(n-1)^{\frac{1-p}{2|\beta|}+\frac{\nu}{2}+n-\frac{7}{4}}}{(\nu+n)^{(\nu+n)-1/2}}, \qquad n \gg 1.$$

Thus, we would like to test $\lim_{n\to\infty} \log(d_{n+1}/d_n) < 0$, where $n \gg 1$. First observe that

$$\begin{aligned}\log\Big(\frac{d_{n+1}}{d_n}\Big) &= \Big(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{3}{4}\Big)\log(n) - \Big(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{7}{4}\Big)\log(n-1) \\ &\quad + \Big(\nu + n - \frac{1}{2}\Big)\log(\nu+n) - \Big(\nu + n + \frac{1}{2}\Big)\log(\nu+n+1).\end{aligned}$$

Then, making use of the approximation $\log(n+a) \approx \log(n) + a/n - a^2/(2n^2) + \cdots$, we find

$$\lim_{n\to\infty} \log\Big(\frac{d_{n+1}}{d_n}\Big) = \lim_{n\to\infty} -\frac{1}{n}\Big(\frac{1-p}{2|\beta|} + \frac{\nu}{2} + n - \frac{7}{4}\Big) + \Big(\nu + n - \frac{1}{2}\Big)\frac{\nu}{n} - \Big(\nu + n + \frac{1}{2}\Big)\frac{\nu+1}{n} = -2,$$

which concludes the proof.

B.3 Proof of Proposition 5.3

Observe that from Lemma 4.1 we obtain,

$$\begin{aligned}&\int_0^t \mathbb{E}_x\left[\mathbb{I}_{\{\zeta^\phi > u\}} \int_{\mathbb{R}} \log^2\Big(1 + \frac{y}{X_{u-}^\phi}\Big)\pi^{\phi,1}(X_{u-}^\phi, y)dy\right] du \\ &= \int_{(0,\infty)} \int_0^t \sum_{n=1}^\infty e^{-\lambda_n s}\Big\{\Big(\int_{E\setminus\{x\}} \log^2(y)\psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}_u^{\phi,1}\psi_n(x)\big] \\ &\quad - 2\Big(\int_{E\setminus\{x\}} \log(y)\psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}_u^{\phi,1}\big(\log(x)\psi_n(x)\big)\big] \\ &\quad + \Big(\int_{E\setminus\{x\}} \psi_n(y)\mathfrak{m}(y)dy\Big)\big[\mathcal{P}_u^{\phi,1}\big(\log^2(x)\psi_n(x)\big)\big]\Big\} du\,\nu(ds), \qquad \text{(B.2)}\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}_u^{\phi,1}\psi_n(x) &= \sum_{m=1}^\infty e^{-\phi(\lambda_m)u}(\psi_n,\psi_m)\psi_m(x) = e^{-\phi(\lambda_n)u}\psi_n(x), \\ \mathcal{P}_u^{\phi,1}\big(\log^\delta(x)\psi_n(x)\big) &= \sum_{m=1}^\infty e^{-\phi(\lambda_m)u} d_{n,m}^\delta \psi_m(x), \quad d_{n,m}^\delta = \int_E \log^\delta(y)\psi_n(y)\psi_m(y)\mathfrak{m}(y)dy, \quad \delta \in \{1,2\}.\end{aligned}$$

Explicit representation of $d_{n,m}^1$ and $d_{n,m}^2$ are found using

$$d_{n,m}^\delta = 2|\beta|A^{\nu+1}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E y^{2c-2\beta}\log^\delta(y)e^{-Ay^{-2\beta}}L_{n-1}^\nu(Ay^{-2\beta})L_{m-1}^\nu(Ay^{-2\beta})dy, \quad \delta \in \{1,2\}.$$

Making the change of variable $z = Ay^{-2\beta}$ we obtain

$$\begin{aligned} d^1_{n,m} &= \frac{1}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu(\log(z)-\log(A))e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz \\ &= \frac{1}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu\log(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz \\ &\quad - \frac{\log(A)}{2|\beta|}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz, \end{aligned}$$

and

$$\begin{aligned} d^2_{n,m} &= \frac{1}{(2|\beta|)^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu(\log(z)-\log(A))^2e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz \\ &= \frac{1}{4|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu\log^2(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz \\ &\quad - \frac{\log(A)}{2|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu\log(z)e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz \\ &\quad + \frac{\log^2(A)}{4|\beta|^2}\sqrt{\frac{(n-1)!(m-1)!}{\Gamma(\nu+n)\Gamma(\nu+m)}}\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz. \end{aligned}$$

Define the functions Θ^δ_n by the integral

$$\Theta^\delta_n(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}\log^\delta(z)L^\nu_n(z)dz, \qquad \delta\in\{1,2\}, \qquad \mathrm{Re}(\alpha)>0. \tag{B.3}$$

Then, using the identity $\int_E z^\nu e^{-z}L^\nu_{n-1}(z)L^\nu_{m-1}(z)dz = \delta_{n,m}\Gamma(\nu+n)/(n-1)!$ (where $\delta_{n,m}$ is the Kronecker delta) and the series expansion for the Generalized Laguerre polynomials ($L^\nu_n(z) = \Gamma(\nu+n+1)/n!\sum_{k=0}^n(-n)_kz^k/(\Gamma(\nu+k+1)k!)$) to obtain

$$d^1_{n,m} = \frac{1}{2|\beta|}\sqrt{\frac{(m-1)!\Gamma(\nu+n)}{(n-1)!\Gamma(\nu+m)}}\left\{\sum_{k=0}^{n-1}\frac{(1-n)_k\Theta^1_{m-1}(\nu+k+1)}{\Gamma(\nu+k+1)k!} - \log(A)\delta_{n,m}\right\},$$

and

$$d^2_{n,m} = \frac{1}{4|\beta|^2}\sqrt{\frac{(m-1)!\Gamma(\nu+n)}{(n-1)!\Gamma(\nu+m)}}\left\{\sum_{k=0}^{n-1}\frac{(1-n)_k\big(\Theta^2_{m-1}(\nu+k+1)-2\log(A)\Theta^1_{m-1}(\nu+k+1)\big)}{\Gamma(\nu+k+1)k!} + \log^2(A)\delta_{n,m}\right\}.$$

Similarly, we obtain

$$c_n = \int_E\psi_n(y)\mathfrak{m}(y)dy = \frac{A^{\frac{1-2c}{4|\beta|}}(1/(2|\beta|))_{n-1}\,\Gamma(c/|\beta|+1)}{\sqrt{(n-1)!|\mu+b|\Gamma(\nu+n)}},$$

where c_n is found by setting $p=0$ in (5.4). Also,

$$\begin{aligned} \int_E\log(y)\psi_n(y)\mathfrak{m}(y)dy &= \frac{A^{\frac{1-2c}{4|\beta|}}}{2|\beta|}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}(\log(z)-\log(A))e^{-z}L^\nu_{n-1}(z)dz \\ &= \frac{A^{\frac{1-2c}{4|\beta|}}}{2|\beta|}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\Theta^1_{n-1}\Big(\frac{c+|\beta|}{|\beta|}\Big) - \frac{\log(A)}{2|\beta|}c_n, \end{aligned}$$

and

$$\int_E \log^2(y)\psi_n(y)\mathfrak{m}(y)dy = \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\int_E z^{\frac{c+|\beta|}{|\beta|}-1}(\log(z)-\log(A))^2 e^{-z}L_{n-1}^{\nu}(z)dz$$
$$= \frac{A^{\frac{1-2c}{4|\beta|}}}{4|\beta|^2}\sqrt{\frac{(n-1)!}{|\mu+b|\Gamma(\nu+n)}}\Big(\Theta_{n-1}^2\Big(\frac{c+|\beta|}{|\beta|}\Big) - 2\log(A)\Theta_{n-1}^1\Big(\frac{c+|\beta|}{|\beta|}\Big)\Big) + \frac{\log^2(A)c_n}{4|\beta|^2}.$$

Substituting the above expressions into Eq. (B.2) and using the relation (B.3) for Θ_n^1, and Θ_n^2 which can be obtained from the integrals 2.19.6.1-2.19.6.4, in Prudnikov et al. (1986), p.469, we arrive to the final expression (5.6).

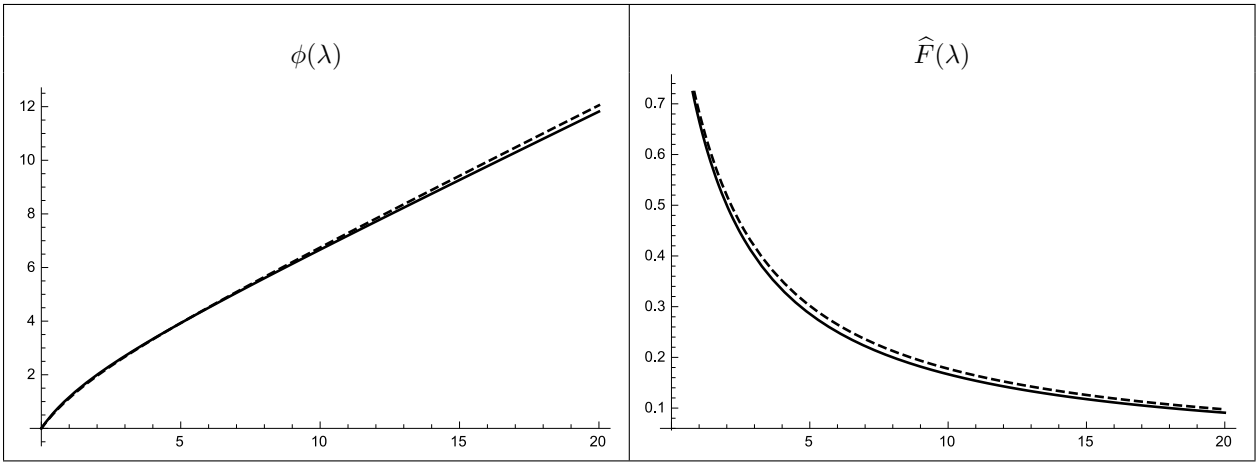

Figure 1: We plot $\phi_0(\lambda)$ (solid) and $\phi^*(\lambda)$ (dashed) from equation (7.7), as well as $\widehat{F}_0(\lambda)$ (solid) and $\widehat{F}^*(\lambda)$ (dashed) from equation (7.8).

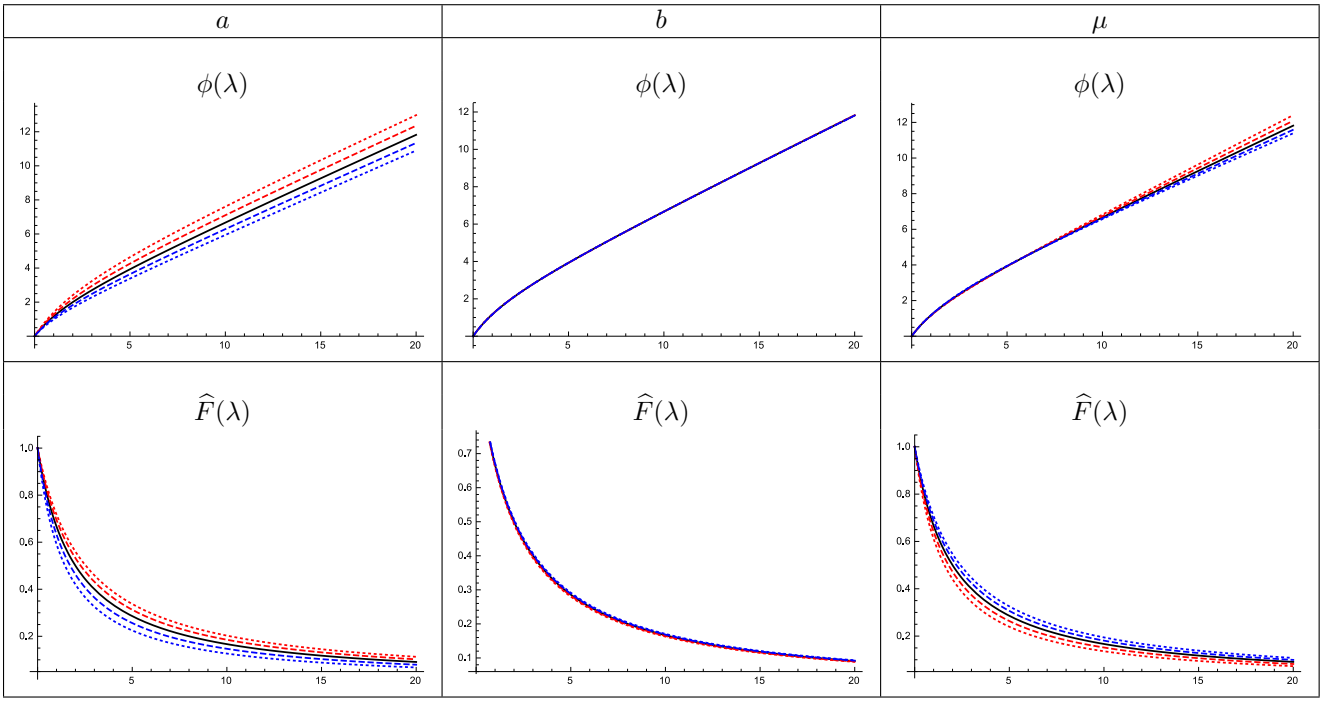

Figure 2: These above plots demonstrate how misspecification of one of the parameters (a_0, b_0, or μ_0) of the JDCEV process X affect the ϕ^* and $\widehat{F}^*$, the estimates of ϕ and F. We fix $\chi_0 \in \{a_0, b_0, \mu_0\}$ and perturb it according to $\chi_0 \to \chi_\varepsilon = (1+\varepsilon)\chi_0$. We then obtain ϕ^* according to steps 4 and 5 of section 7.1. In each of the plots the (red-dotted, red-dashed, black, blue-dashed, blue-dotted) lines correspond to $\varepsilon = (-0.1, -0.05, 0, 0.05, 0.1)$ respectively. We omit the plots for $\chi_0 \in \{c_0, \beta_0\}$ since, in our numerical experiments, these parameters do not affect ϕ^* as drastically as $\{a_0, b_0, \mu_0\}$.

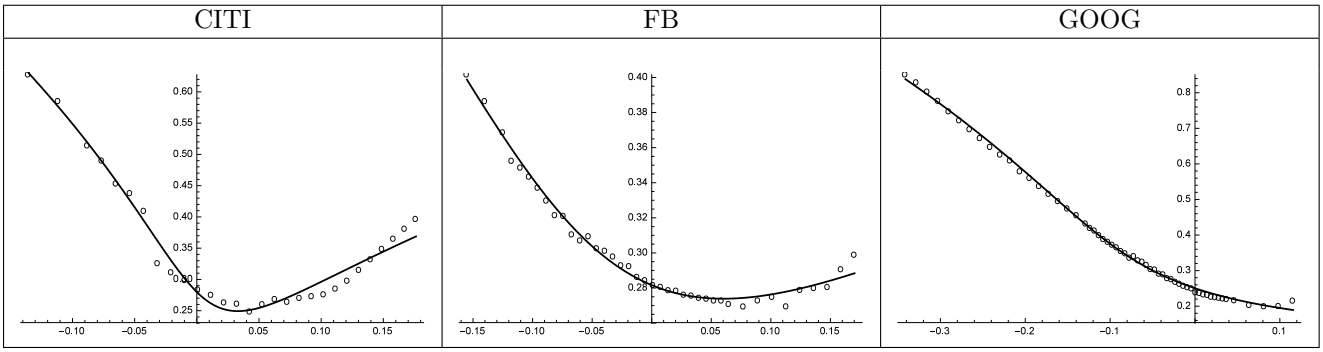

Figure 3: We plot the raw implied volatility data and SVI fit for three stocks (CITI, FB and GOOG). Units of the horizontal axis are log-moneyness (i.e. log strike divided by log spot). The time to maturity in all three plots is $t = 1/12$ years.

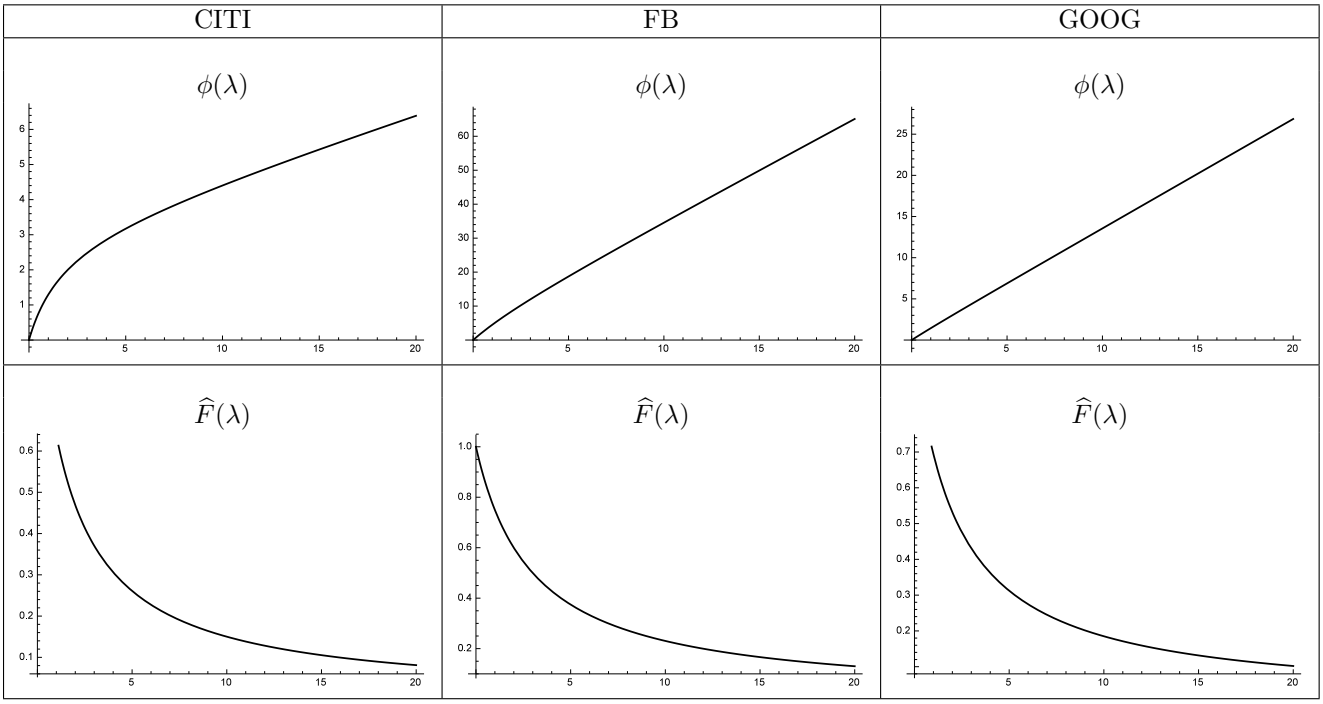

Figure 4: We plot the nonparametric estimates ϕ^* and $\widehat{F}$ for the three stocks considered in Section 7.3.

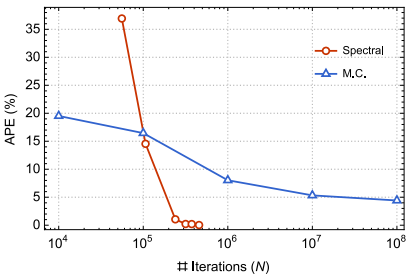

Figure 5: We plot the absolute relative error APE (%) as a function of the number of iterations used to compute K_{var}.

References

Abate, J. and W. Whitt (2006). A Unified Framework for Numerically Inverting Laplace Transforms. *INFORMS Journal on Computing 18*(4), 408–421.

Barndorff-Nielsen, O. E. and N. Shephard (2002). Estimating Quadratic Variation Using Realized Variance. *Journal of Applied Econometrics 17*(5), 457–477.

Bertoin, J. (1996). *Lévy Processes*, Volume 121 of *Cambridge Tracts in Mathematics*. Cambridge: Cambridge University Press.

Bertoin, J. (2004). Subordinators: Examples and Applications. In P. Bernard (Ed.), *Lectures on Probability Theory and Statistics Ecole d'Ete de Probabilites de Saint-Flour XXVII - 1997*, Volume 1717 of *Lecture Notes in Mathematics: Mathematics and Statistics*, pp. 1–97. Springer Berlin / Heidelberg.

Bochner, S. (1949). Diffusion Equation and Stochastic Processes. In *Proceedings of the National Academy of Sciences*, Volume 35 (7), pp. 368–370.

Bondarenko, O. (2003). Estimation of Risk-Neutral Densities Using Positive Convolution Approximation. *Journal of Econometrics 116*(1-2), 85–112.

Borodin, A. and P. Salminen (2002). *Handbook of Brownian Motion: Facts and Formulae*. Probability and Its Applications. Basel.: Birkhauser Verlag AG, 2 Rev. Ed.

Breeden, D. T. and R. H. Litzenberger (1978). Prices of State-Contingent Claims Implicit in Option Prices. *The Journal of Business 51*(4), 621–651.

Carr, P., R. Lee, and L. Wu (2012). Variance Swaps on Time-Changed Lévy Processes. *Finance and Stochastics 16*(2), 335–355.

Carr, P. and V. Linetsky (2006). A Jump to Default Extended CEV Model: An Application of Bessel Processes. *Finance and Stochastics 10*(3), 303–330.

Carr, P. and W. Schoutens (2008). Hedging under the Heston Model with Jump-to-Default. *International Journal of Theoretical & Applied Finance 11*(4), 403–414.

Cont, R. and P. Tankov (2004). *Financial Modelling with Jump Processes*. Boca Raton: CRC Press.

Cox, J. C. (1975). Notes on Option Pricing I: Constant Elasticity of Variance Diffusions. *Reprinted in The Journal of Portfolio Management, December 1996 23*, 15–17.

Davies, E. B. (2007). *Linear Operators and Their Spectra*. Number 106 in Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press.

Feng, L. and V. Linetsky (2008). Pricing Options in Jump-Diffusion Models: An Extrapolation Approach. *Operations Research 56*(2), 304–325.

Gatheral, J. (2004). A Parsimonious Arbitrage-Free Implied Volatility Parameterization with Application to the Valuation of Volatility Derivatives. Presentation at Global Derivatives & Risk Management, Madrid.

Gatheral, J. and A. Jacquier (2012, April). Arbitrage-Free SVI Volatility Surfaces. *Working paper*.

Jeanblanc, M., M. Yor, and M. Chesney (2009). *Mathematical Methods for Financial Markets*. Springer Finance. London, UK: Springer.

Lewis, A. L. (1998). Applications of Eigenfunction Expansions in Continuous-Time Finance. *Mathematical Finance 8*(4), 349–383.

Li, L. and V. Linetsky (2013a, August). Discretely Monitored First Passage Problems: An Eigenfunction Expansion Approach. Submitted for publication.

Li, L. and V. Linetsky (2013b). Optimal Stopping and Early Exercise: An Eigenfunction Expansion Approach. *Operations Research 61*(3), 625–644.

Li, L., R. Mendoza-Arriaga, and D. Mitchell (2015). Analytical Representations for the Basic Affine Jump Diffusion. *Submitted*.

Lindsay, A. and D. Brecher (2012). Simulation of the CEV Process and the Local Martingale Property. *Mathematics and Computers in Simulation 82*(5), 868–878.

Linetsky, V. (2004). The Spectral Decomposition of the Option Value. *International Journal of Theoretical and Applied Finance 7*(3), 337–384.

Linetsky, V. (2008). Spectral Methods in Derivatives Pricing. In J. Birge and V. Linetsky (Eds.), *Handbooks in Operations Research and Management Science: Financial Engineering*, Volume 15, pp. 223–300. Amsterdam: Elsevier.

Makarov, R. N. and D. Glew (2010). Exact Simulation of Bessel Diffusions. *Monte Carlo Methods and Applications 16*(3-4), 283–306.

McKean, H. P. (1956). Elementary Solutions for Certain Parabolic Partial Differential Equations. *Transactions of the American Mathematical Society 82*(2), 519–548.

Mendoza-Arriaga, R. (2009). *Unified Credit-Equity Modeling*. Ph. D. thesis, Northwestern University, IEMS., Evanston, IL.

Mendoza-Arriaga, R., P. Carr, and V. Linetsky (2010). Time Changed Markov Processes in Credit-Equity Modeling. *Mathematical Finance 20*(4), 527–569.

Mendoza-Arriaga, R. and V. Linetsky (2014a). Multivariate Subordination of Markov Processes with Financial Applications. *To appear* Mathematical Finance.

Mendoza-Arriaga, R. and V. Linetsky (2014b). Time-Changed CIR Default Intensities with Two-Sided Mean-Reverting Jumps. *The Annals of Applied Probability 24*(2), 811–856.

Nikiforov, A. and V. Uvarov (1988). *Special Functions of Mathematical Physics: A Unified Introduction with Applications*. Basel: Birkhäuser.

Phillips, R. S. (1952). On the Generation of Semigroups of Linear Operators. *Pacific Journal of Mathematics 2*(3), 343–369.

Prudnikov, A. P., Y. A. Brychkov, and O. I. Marichev (1986). *Integrals Series: Special Functions*, Volume II of *Integrals and Series*. New York: Gordon and Breach Science Publishers.

Prudnikov, A. P., Y. A. Brychkov, and O. I. Marichev (1990). *Integrals Series: More Special Functions*, Volume III of *Integrals and Series*. New York: Gordon and Breach Science Publishers.

Reed, M. and B. Simon (1980). *Methods of Modern Mathematical Physics* (Revised ed.), Volume I: Functional Analysis. San Diego: Academic Press.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge.: Cambridge University Press.